

PAPER

Universal relations for a spin-polarized Fermi gas in two dimensions

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Universal relations for a spin-polarized Fermi gas in two dimensions

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Abstract

We derive the full set of universal relations for spin-polarized Fermi gases with p -wave interaction in two dimensions, simply using the short-range asymptotic behavior of fermion-pair wave functions. For p -wave interactions, an additional contact related to the effective range needs to be introduced, besides the one related to the scattering volume. Since the subleading tail (k^{-4}) of the large-momentum distribution cannot fully be captured by the contacts defined by the adiabatic relations, an extra term resulted from the center-of-mass motions of the pairs gives rise to an additional divergence in the kinetic energy of the system, besides those related to the contacts defined. We show in Tan's energy theorem that if only two-body correlations are taken into account, all these divergences are reasonably removed, leading to a finite internal energy of the system. In addition, we find that all the other universal relations, such as the high-frequency behavior of the radio-frequency response, short-range behavior of the pair correlation function, generalized virial theorem, and pressure relation, remain unaffected by the center-of-mass motions of the pairs, and are fully governed by the contacts defined by the adiabatic relations. Our results confirm the feasibility of generalizing the contact theory for higher-partial-wave scatterings, and could readily be confirmed in current experiments with ultracold ^{40}K and ^6Li atoms.

Keywords: Fermi gases, Tan's contact theory, p -wave interactions

1. Introduction

In the past decades, ultracold Fermi gases with short-range interactions have attracted a great deal of interest due to their unique properties [1, 2]. Especially, near scattering resonances, where the scattering length a is much larger than all the other length scales, such systems

manifest *universality*: the many-body properties at long distance are primarily determined by a , and become irrelevant to the specific form of the short-range interatomic interactions [3]. For strongly interacting two-component Fermi gases with s -wave interactions, a set of universal relations that follow from the short-range behavior of the simple two-body physics were derived by Shina Tan, governing the key properties of many-body systems [4]. Afterwards, more universal relations were obtained [5]. All these relations are characterized by the only universal quantity named *contact*, and then the concept of contact becomes significantly important in ultracold atoms both theoretically and experimentally [6–17].

However, for higher partial waves, Tan’s universal relations should be amended, since the short-distance behavior of interatomic interactions cannot simply be characterized by a single scattering parameter. More microscopic parameters need to be involved besides the scattering length (or scattering volume, or some quantity like that), such as the effective range, which may result in non-trivial corrections. As the simplest case of higher partial wave scatterings, the p -wave many-body systems have attracted both experimental and theoretical attention [18–29]. Considering the finite-range effect, more contacts are needed when generalizing the s -wave contact theory to the p -wave case [30–33]. It is found that if one tries to define the contacts according to the adiabatic relations, the subleading tail of the large-momentum distribution cannot fully be captured, and an extra term appears, due to the center-of-mass (c.m.) motions of Cooper pairs [32]. This is a general feature of strongly interacting Fermi gases near p -wave resonances. Very recently, the p -wave contacts defined by the adiabatic relations for a two-dimensional (2D) Fermi gas are discussed in [35], and even the three-body contact is introduced in [36] when taking the super Efimov effect into account. However, the full set of universal relations still lack justification, such as the energy theorem, the short-distance behavior of the pair correlation function, and so on.

In this paper, we systematically study the full set of the p -wave universal relations, choosing the 2D spin-polarized Fermi gas as the model system. We present a derivation of the universal relations following the route of Tan’s original work about the s -wave case, in which only two-body correlations are taken into account [4]. Among these universal relations for p -wave interactions, the energy theorem is of particular interest, since it is directly related to the feasibility of the contact interaction. For s -wave interactions, the kinetic and interaction energies are both ultraviolet divergent in the zero-range limit, but these divergences cancel with each other when they add up. The resulting internal energy of the system remains physically finite, which can be expressed using Tan’s energy theorem, involving only the momentum distribution and the contact. Here, we show that the internal energy as a functional of the momentum distribution still exists for a spin-polarized Fermi gas near p -wave resonances in two dimensions, and thereby establish Tan’s energy theorem for p -wave interactions. The derivation of the p -wave energy theorem is nontrivial. Unlike the s -wave case, the many-body wave function of a p -wave system may not be well normalized in the zero-range limit. It actually diverges logarithmically as the interaction range vanishes. Starting from the short-range behavior of the many-body wave function, we define the p -wave contacts according to the adiabatic relations, and then verify the behavior of the momentum distribution at large k [35]. We find that both the leading (k^{-2}) and subleading (k^{-4}) tails give rise to the ultraviolet divergence for the kinetic energy. While the subleading tail cannot fully be described by the contacts defined by the adiabatic relations, an additional divergence for the kinetic energy arises due to the c.m. motions of the pairs, besides those related to the contacts. Here, we demonstrate that all these divergences can reasonably be removed, leading to a well-defined internal energy of the system.

The high-frequency tail of the radio-frequency (RF) response of the system is also governed by the contacts, and is experimentally used as a way of measuring the contacts. It links to the momentum distribution $n(\mathbf{k})$ as $\sum_{\mathbf{k}} n(\mathbf{k}) \delta(\hbar\omega - \hbar^2 k^2/M)$, as the RF $\omega \rightarrow \infty$, a result first derived by Schneider and Randeria according to the properties of the spectral function [9]. Here \hbar is the Planck's constant and M is the atomic mass. At first glance, the c.m. contribution of the pairs in the subleading k^{-4} tail of the momentum distribution $n(\mathbf{k})$ should be involved in the asymptotic behavior of the RF response at high frequencies. However, after a rigorous calculation according to the Fermi's golden rule, we find that the high-frequency tail of the RF response is determined by $\sum_{\mathbf{k}} n'(\mathbf{k}) \delta(\hbar\omega - \Delta E)$, where ΔE is the energy difference between the final state after the RF transition and the initial state, and $n'(\mathbf{k})$ is not exactly the momentum distribution of the system (see equation (69)). $n'(\mathbf{k})$ has the same leading behavior as that of the momentum distribution $n(\mathbf{k})$, but different subleading behavior, in which the c.m. contribution is excluded. After carefully dealing with this, we finally discover that the high-frequency tail of the RF response is fully described by the contacts defined by the adiabatic relations.

In addition, we also obtain the short-distance behavior of the pair correlation function, which is determined merely by the short-range behavior of the relative motions of the pairs. Naturally, it is fully captured by the contacts we defined. Finally, we derive the generalized virial theorem as well as the pressure relation. These thermodynamic relations are easily derived by using the adiabatic relations, and obviously, can be fully described by the contacts defined by the adiabatic relations.

This paper is arranged as follows. In section 2, we present the definitions of the p -wave contacts, and derive the specific form of the adiabatic relations for a 2D spin-polarized Fermi gas. The asymptotic behavior of the momentum distribution at large momentum is discussed in section 3. In section 4, we derive the Tan's energy theorem for p -wave interactions, in which the internal energy of the system is expressed as a functional of the momentum distribution, and demonstrate how all the divergences are removed. The high-frequency behavior of the RF response of the system is studied in section 5 according to the Fermi's golden rule, and in section 6, the short-distance behavior of pair correlation function is obtained. The general virial theorem is acquired by using the adiabatic relations as well as the pressure relation in section 7. Finally, our main results are summarized in section 8.

2. Adiabatic relations

Let us consider a strongly interacting spin-polarized Fermi gas with total particle number N in two dimensions. The interatomic collision is dominated by the p -wave interaction with a short range ϵ , which is much smaller than all the other length scales of the system. Then we may deal with the interaction by setting a short-range boundary condition on many-body wave functions: when any two of fermions, for example, i and j , get close to each other, a many-body wave function in two dimensions can be written as

$$\Psi_{2D}(\mathbf{X}, \mathbf{R}, \mathbf{r}) = \sum_{\sigma=\pm} \mathcal{A}_{\sigma}(\mathbf{X}, \mathbf{R}) \psi_{\sigma}(\mathbf{r}), \quad (1)$$

where $\mathbf{r} = \mathbf{r}_i - \mathbf{r}_j$, $\mathbf{R} = (\mathbf{r}_i + \mathbf{r}_j)/2$ are, respectively, the relative and c.m. coordinates of the pair (i, j) , \mathbf{X} includes the degrees of freedom of all the other fermions, and the index $\sigma = \pm$ denotes two different magnetic components of the p -wave wave function. The function $\mathcal{A}_{\sigma}(\mathbf{X}, \mathbf{R})$ is regular and $\psi_{\sigma}(\mathbf{r})$ is the two-body wave function describing the relative motion of the pair, which should take the form (unnormalized)

$$\psi_\sigma(\mathbf{r}) = \frac{\pi}{2} q [J_1(qr) \cot \delta_\sigma - N_1(qr)] \Omega_1^{(\sigma)}(\varphi) \quad (2)$$

outside the range of the interatomic interaction, where $J_\nu(\cdot)$, $N_\nu(\cdot)$ are the Bessel functions of the first and second kinds, q is the relative wave number of the pair, $\Omega_m^{(\sigma)}(\varphi) \equiv e^{i\sigma m\varphi}/\sqrt{2\pi}$ is the angular function with respect to the azimuthal angle φ of the vector \mathbf{r} . We note that the *regularity* of the function $\mathcal{A}_\sigma(\mathbf{X}, \mathbf{R})$ implies that no more pairs except the fermions i and j can interact with each other, because of such a short-range interaction.

The interactions between spin-polarized fermions can be tuned using p -wave Feshbach resonances experimentally, and Tan's adiabatic relations state how the total energy of the system changes accordingly when the interatomic interaction is adiabatically adjusted. To derive the adiabatic relations, we consider two many-body wave functions Ψ_{2D} and Ψ'_{2D} corresponding to different interaction strengths, and they should satisfy the Schrödinger equations with different energies

$$\sum_{i=1}^N \left[-\frac{\hbar^2}{2M} \nabla_i^2 + U(\mathbf{r}_i) \right] \Psi_{2D} = E \Psi_{2D}, \quad (3)$$

$$\sum_{i=1}^N \left[-\frac{\hbar^2}{2M} \nabla_i^2 + U(\mathbf{r}_i) \right] \Psi'_{2D} = E' \Psi'_{2D}, \quad (4)$$

if there is no pair of fermions within the range of the interaction. Here, M is the atomic mass and $U(\mathbf{r}_i)$ is the external potential experienced by the i th fermion. Then it follows from equations (3) and (4) that [32]

$$\begin{aligned} (E - E') \int_{\mathcal{S}_\epsilon} \prod_{i=1}^N d\mathbf{r}_i \Psi'_{2D}{}^* \Psi_{2D} \\ = -\frac{\hbar^2}{M} \mathcal{N} \oint_{r=\epsilon} (\Psi'_{2D}{}^* \nabla_{\mathbf{r}} \Psi_{2D} - \Psi_{2D} \nabla_{\mathbf{r}} \Psi'_{2D}{}^*) \cdot \hat{\mathbf{n}} dl, \end{aligned} \quad (5)$$

where $\mathcal{N} = N(N-1)/2$ is the number of all the possible ways to pair atoms, the domain \mathcal{S}_ϵ is the set of all configurations $(\mathbf{r}_i, \mathbf{r}_j)$, in which $r = |\mathbf{r}_i - \mathbf{r}_j| > \epsilon$, l is the boundary of \mathcal{S}_ϵ that the distance between the two fermions in the pair (i, j) is ϵ , and $\hat{\mathbf{n}}$ is the direction normal to l , but is opposite to the radial direction. Expanding the many-body wave function (1) at small r , we obtain

$$\begin{aligned} \Psi_{2D}(\mathbf{X}, \mathbf{R}, \mathbf{r}) \approx \sum_{\sigma} \mathcal{A}_{\sigma}(\mathbf{X}, \mathbf{R}) \left[\frac{1}{r} - \frac{q^2}{2} r \ln \frac{r}{2b_{\sigma}} \right. \\ \left. - \left(\frac{\pi}{4a_{\sigma}} + \frac{2\gamma - 1}{4} q^2 \right) r + \mathcal{O}(r^3) \right] \Omega_1^{(\sigma)}(\varphi), \end{aligned} \quad (6)$$

where γ is the Euler's constant, and we have used the effective-range expansion of the p -wave scattering phase shift for 2D systems [34], i.e.

$$\cot \delta_{\sigma} = -\frac{1}{a_{\sigma} q^2} + \frac{2}{\pi} \ln(qb_{\sigma}), \quad (7)$$

and a_{σ}, b_{σ} are the scattering area and effective range, respectively, with the dimensions of length² and length¹. We should note that the pair relative wave number q is generally dependent

on \mathbf{X} as well as \mathbf{R} , due to the external confinement U and the interatomic interactions, and can formally be written as [12, 32]

$$\frac{\hbar^2 q^2}{M} = E - \frac{1}{\mathcal{A}_\sigma(\mathbf{X}, \mathbf{R})} [T(\mathbf{X}, \mathbf{R}) + U(\mathbf{X}, \mathbf{R})] \mathcal{A}_\sigma(\mathbf{X}, \mathbf{R}), \quad (8)$$

and $T(\mathbf{X}, \mathbf{R})$ and $U(\mathbf{X}, \mathbf{R})$ are respectively the kinetic and external potential operators including the c.m. motion of the pair (i, j) and those of the rest of the fermions. Inserting the asymptotic form of the many-body wave function (6) into equation (5), and letting $E' \rightarrow E$, $a'_\sigma \rightarrow a_\sigma$, $b'_\sigma \rightarrow b_\sigma$, we easily obtain

$$\begin{aligned} \delta E \cdot \int_{S_\epsilon} \prod_{i=1}^N d\mathbf{r}_i |\Psi_{2D}|^2 &= \sum_\sigma \left[-\frac{\pi \hbar^2}{2M} \mathcal{N} \mathcal{I}_a^{(\sigma)} \cdot \delta a_\sigma^{-1} \right. \\ &\quad \left. + \mathcal{N} \mathcal{E}_\sigma \cdot \delta \ln b_\sigma + \mathcal{N} \mathcal{I}_a^{(\sigma)} \left(\ln \frac{2b_\sigma}{\epsilon} - \gamma \right) \cdot \delta E \right], \end{aligned} \quad (9)$$

where

$$\mathcal{I}_a^{(\sigma)} \equiv \int d\mathbf{X} d\mathbf{R} |\mathcal{A}_\sigma(\mathbf{X}, \mathbf{R})|^2, \quad (10)$$

$$\mathcal{E}_\sigma \equiv \int d\mathbf{X} d\mathbf{R} \mathcal{A}_\sigma^*(E - T - U) \mathcal{A}_\sigma. \quad (11)$$

Using the normalization of the wave function (see appendix A)

$$\int_{S_\epsilon} \prod_{i=1}^N d\mathbf{r}_i |\Psi_{2D}|^2 = 1 + \mathcal{N} \sum_\sigma \mathcal{I}_a^{(\sigma)} \left(\ln \frac{2b_\sigma}{\epsilon} - \gamma \right). \quad (12)$$

Equation (9) can further be simplified as

$$\delta E = \sum_\sigma \left[-\frac{\pi \hbar^2}{2M} \mathcal{N} \mathcal{I}_a^{(\sigma)} \cdot \delta a_\sigma^{-1} + \mathcal{N} \mathcal{E}_\sigma \cdot \delta \ln b_\sigma \right], \quad (13)$$

which yields

$$\frac{\partial E}{\partial a_\sigma^{-1}} = -\frac{\pi \hbar^2}{2M} \mathcal{N} \mathcal{I}_a^{(\sigma)}, \quad (14)$$

$$\frac{\partial E}{\partial \ln b_\sigma} = \mathcal{N} \mathcal{E}_\sigma. \quad (15)$$

3. Tail of the momentum distribution at large k and contacts

In this section, we are going to study the asymptotic behavior of the large momentum distribution for a spin-polarized Fermi gas. The momentum distribution of the i th fermion is defined as

$$n_i(\mathbf{k}) \equiv \int \prod_{t \neq i} d\mathbf{r}_t \left| \tilde{\Psi}_i(\mathbf{k}) \right|^2, \quad (16)$$

where $\tilde{\Psi}_i(\mathbf{k}) \equiv \int d\mathbf{r}_i \Psi_{2D} e^{-i\mathbf{k}\cdot\mathbf{r}_i}$, and then the total momentum distribution is $n(\mathbf{k}) = \sum_{i=1}^N n_i(\mathbf{k})$. When the pair (i, j) get close but still outside the interaction range, i.e. $r(\sim 0^+) > \epsilon$, while all the other fermions are far away, we may again expand the many-body wave function Ψ_{2D} (1) at $\mathbf{r} \approx 0$, and rewrite it as the following ansatz

$$\Psi_{2D}(\mathbf{X}, \mathbf{R}, \mathbf{r}) = \sum_{\sigma} \left[\frac{\mathcal{A}_{\sigma}(\mathbf{X}, \mathbf{R})}{r} + \mathcal{B}_{\sigma}(\mathbf{X}, \mathbf{R}) r \ln r + \mathcal{C}_{\sigma}(\mathbf{X}, \mathbf{R}) r \right] \Omega_1^{(\sigma)}(\varphi) + \mathbf{r} \cdot \mathbf{L}(\mathbf{X}, \mathbf{R}) + \mathcal{O}(r^3), \quad (17)$$

where \mathcal{A}_{σ} , \mathcal{B}_{σ} , \mathcal{C}_{σ} and \mathbf{L} are all regular functions, and the term $\mathbf{r} \cdot \mathbf{L}(\mathbf{X}, \mathbf{R})$ represents the coupling between the relative and c.m. motions of the pair (i, j) , resulted from the external confinement. Comparing equations (6) and (17) at small r , we find

$$\mathcal{B}_{\sigma} = -\frac{q^2}{2} \mathcal{A}_{\sigma}, \quad (18)$$

$$\begin{aligned} \mathcal{C}_{\sigma} &= \left(\frac{1-2\gamma}{4} q^2 - \frac{\pi}{4a_{\sigma}} + \frac{q^2}{2} \ln 2b_{\sigma} \right) \mathcal{A}_{\sigma} \\ &= -\left(\frac{1-2\gamma}{2} + \ln 2b_{\sigma} \right) \mathcal{B}_{\sigma} - \frac{\pi}{4a_{\sigma}} \mathcal{A}_{\sigma}, \end{aligned} \quad (19)$$

and \mathcal{E}_{σ} defined in equation (11) can alternatively be rewritten as

$$\mathcal{E}_{\sigma} = -\frac{2\hbar^2}{M} \int d\mathbf{X} d\mathbf{R} \mathcal{A}_{\sigma}^*(\mathbf{X}, \mathbf{R}) \mathcal{B}_{\sigma}(\mathbf{X}, \mathbf{R}) \equiv -\frac{2\hbar^2}{M} \mathcal{I}_b^{(\sigma)}, \quad (20)$$

where

$$\mathcal{I}_b^{(\sigma)} \equiv \int d\mathbf{X} d\mathbf{R} \mathcal{A}_{\sigma}^*(\mathbf{X}, \mathbf{R}) \mathcal{B}_{\sigma}(\mathbf{X}, \mathbf{R}) \quad (21)$$

is obviously real. The asymptotic behavior of the momentum distribution at large \mathbf{k} but still smaller than ϵ^{-1} is determined by that of the wave function at short distance, then we have

$$\tilde{\Psi}_i(\mathbf{k}) \underset{k \rightarrow \infty}{\approx} \sum_{j \neq i} e^{-i\mathbf{k}\cdot\mathbf{r}_j} \int d\mathbf{r} \Psi_{2D} \left(\mathbf{X}, \mathbf{r}_j + \frac{\mathbf{r}}{2}, \mathbf{r} \right) e^{-i\mathbf{k}\cdot\mathbf{r}}. \quad (22)$$

With the help of the plane-wave expansion

$$e^{i\mathbf{k}\cdot\mathbf{r}} = \sqrt{2\pi} \sum_{m=0}^{\infty} \sum_{\sigma=\pm} \eta_m i^m J_m(kr) e^{-i\sigma m \varphi_{\mathbf{k}}} \Omega_m^{(\sigma)}(\varphi), \quad (23)$$

where $\eta_m = 1/2$ for $m = 0$, and $\eta_m = 1$ for $m \geq 1$, and $\varphi_{\mathbf{k}}$ is the azimuthal angle of \mathbf{k} , we find

$$\begin{aligned} & \int d\mathbf{r} \frac{\mathcal{A}_{\sigma}(\mathbf{X}, \mathbf{r}_j + \mathbf{r}/2)}{r} \Omega_1^{(\sigma)}(\varphi) e^{-i\mathbf{k}\cdot\mathbf{r}} \\ &= -i\sqrt{2\pi} \mathcal{A}_{\sigma}(\mathbf{X}, \mathbf{r}_j) \frac{e^{i\sigma\varphi_{\mathbf{k}}}}{k} + \sqrt{\frac{\pi}{2}} \alpha_{\sigma}(\mathbf{X}, \mathbf{r}_j, \hat{\mathbf{k}}) \frac{1}{k^2} \\ & \quad + i\frac{\sqrt{2\pi}}{8} \beta_{\sigma}(\mathbf{X}, \mathbf{r}_j, \hat{\mathbf{k}}) \frac{1}{k^3} + \mathcal{O}(k^{-4}), \end{aligned} \quad (24)$$

where

$$\alpha_\sigma(\mathbf{X}, \mathbf{r}_j, \hat{\mathbf{k}}) \equiv k^2 \nabla_{\mathbf{r}_j} \mathcal{A}_\sigma \cdot \nabla_{\mathbf{k}} \frac{e^{i\sigma\varphi_{\mathbf{k}}}}{k}, \quad (25)$$

$$\beta_\sigma(\mathbf{X}, \mathbf{r}_j, \hat{\mathbf{k}}) \equiv k^3 (\nabla_{\mathbf{r}_j} \cdot \nabla_{\mathbf{k}}) \left[\nabla_{\mathbf{r}_j} \mathcal{A}_\sigma \cdot \nabla_{\mathbf{k}} \frac{e^{i\sigma\varphi_{\mathbf{k}}}}{k} \right] \quad (26)$$

only depend on the direction of \mathbf{k} ,

$$\begin{aligned} \int d\mathbf{r} \mathcal{B}_\sigma(\mathbf{X}, \mathbf{r}_j + \frac{\mathbf{r}}{2}) (r \ln r) \Omega_1^{(\sigma)}(\varphi) e^{-i\mathbf{k} \cdot \mathbf{r}} \\ = i2\sqrt{2\pi} \mathcal{B}_\sigma(\mathbf{X}, \mathbf{r}_j) \frac{e^{i\sigma\varphi_{\mathbf{k}}}}{k^3} + \mathcal{O}(k^{-4}), \end{aligned} \quad (27)$$

and

$$\int d\mathbf{r} \mathcal{C}_\sigma(\mathbf{X}, \mathbf{r}_j + \frac{\mathbf{r}}{2}) r \Omega_1^{(\sigma)}(\varphi) e^{-i\mathbf{k} \cdot \mathbf{r}} = 0. \quad (28)$$

In addition, it is obvious that the coupling term $\mathbf{r} \cdot \mathbf{L}(\mathbf{X}, \mathbf{r}_j + \mathbf{r}/2)$ contributes nothing to the tail of the momentum distribution at large k . Therefore, inserting equations (24), (27), and (28) into (22), and then into equation (16), we find the total momentum distribution $n(\mathbf{k})$ at large \mathbf{k} takes the form

$$\begin{aligned} n(\mathbf{k}) \approx \mathcal{N} \int d\mathbf{X} d\mathbf{R} \sum_{\sigma\sigma'} \mathcal{A}_{\sigma'}^* \mathcal{A}_\sigma e^{i(\sigma-\sigma')\varphi_{\mathbf{k}}} \cdot \frac{4\pi}{k^2} + \text{Im} \mathcal{N} \int d\mathbf{X} d\mathbf{R} \sum_{\sigma\sigma'} \mathcal{A}_\sigma e^{i\sigma\varphi_{\mathbf{k}}} \alpha_{\sigma'}^* \cdot \frac{4\pi}{k^3} \\ + \left[-16\pi \text{Re} \mathcal{N} \int d\mathbf{X} d\mathbf{R} \sum_{\sigma\sigma'} \mathcal{A}_{\sigma'}^* \mathcal{B}_\sigma e^{i(\sigma-\sigma')\varphi_{\mathbf{k}}} + \pi \text{Re} \mathcal{N} \int d\mathbf{X} d\mathbf{R} \sum_{\sigma\sigma'} (\alpha_{\sigma'}^* \alpha_\sigma - \mathcal{A}_\sigma e^{i\sigma\varphi_{\mathbf{k}}} \beta_{\sigma'}^*) \right] \frac{1}{k^4} + \mathcal{O}(k^{-5}), \end{aligned} \quad (29)$$

where we have rewritten the integral variable \mathbf{r}_j as \mathbf{R} , and we have also omitted the arguments of the functions \mathcal{A} , \mathcal{B} , α_σ , and β_σ to simplify the expression. If we are only interested in the dependence of the momentum distribution on the amplitude of \mathbf{k} , we may integrate over the direction of \mathbf{k} , and we find all the odd-order terms of k^{-1} vanish. We obtain (see appendix B)

$$n(k) \approx \frac{\sum_\sigma \mathcal{C}_a^{(\sigma)}}{k^2} + \sum_\sigma \left(\mathcal{C}_b^{(\sigma)} + \mathcal{Q}_{\text{cm}}^{(\sigma)} \right) \frac{1}{k^4} + \mathcal{O}(k^{-6}), \quad (30)$$

where the contacts $\mathcal{C}_a^{(\sigma)}$ and $\mathcal{C}_b^{(\sigma)}$ are defined as

$$\mathcal{C}_a^{(\sigma)} \equiv 8\pi^2 \mathcal{N} \mathcal{I}_a^{(\sigma)}, \quad (31)$$

$$\mathcal{C}_b^{(\sigma)} \equiv -32\pi^2 \mathcal{N} \mathcal{I}_b^{(\sigma)}, \quad (32)$$

and

$$\mathcal{Q}_{\text{cm}}^{(\sigma)} \equiv 2\pi^2 \mathcal{N} \int d\mathbf{X} d\mathbf{R} (\nabla_{\mathbf{R}} \mathcal{A}_\sigma^* \cdot \nabla_{\mathbf{R}} \mathcal{A}). \quad (33)$$

Therefore, the adiabatic relations (14) and (15) can alternatively be written as

$$\frac{\partial E}{\partial a_\sigma^{-1}} = -\frac{\hbar^2 \mathcal{C}_a^{(\sigma)}}{16\pi M}, \quad (34)$$

$$\frac{\partial E}{\partial \ln b_\sigma} = \frac{\hbar^2 \mathcal{C}_b^{(\sigma)}}{16\pi^2 M}. \quad (35)$$

We find, similarly as the situation in three dimensions [32], the leading-order term of k^{-2} can be fully described by the contact $\mathcal{C}_a^{(\sigma)}$, while there is an extra term appearing in the subleading-order term of k^{-4} , i.e. $\mathcal{Q}_{\text{cm}}^{(\sigma)}$, in addition to the contact $\mathcal{C}_b^{(\sigma)}$, which results from the c.m. motions of the pairs. We can expect that this additional term should result in significant amendments to the other universal relations.

4. Energy theorem

Because of the short-range p -wave interatomic interactions, the momentum distribution generally decays like k^{-2} at large \mathbf{k} , and subsequently the kinetic energy of the system diverges. Unlike that of the s -wave interaction, the subleading-order term of k^{-4} in the large momentum distribution should also result in an additional divergence of the kinetic energy. In addition, such divergent behavior in the subleading-order term can be fully captured only when both the contact $\mathcal{C}_b^{(\sigma)}$ defined from adiabatic relation (35) and the extra term $\mathcal{Q}_{\text{cm}}^{(\sigma)}$ resulted from the c.m. motions of the pairs are considered. In this section, we show that all of these divergences can be removed, leading to a convergent total internal energy and the p -wave Tan's energy theorem.

In the following, we take only two-body correlations into account, which should be reasonable at two-body resonances, and all higher-order correlations can be neglected. Therefore, in order to avoid the complication of the notations, we first demonstrate the derivation of the energy theorem according to a two-body picture, and then present the general energy theorem for a many-body system. Because only the internal energy of the system is considered, we are going to omit the external confinement, which is trivial to the energy theorem. The Schrödinger equation of two fermions takes the form

$$E\Psi_{2\text{D}} = \left[\sum_{i=1}^2 \left(-\frac{\hbar^2}{2M} \nabla_i^2 \right) + V(\mathbf{r}_1 - \mathbf{r}_2) \right] \Psi_{2\text{D}}, \quad (36)$$

where $V(\mathbf{r}_1 - \mathbf{r}_2)$ is the interatomic interaction with a short range ϵ , out of which we may assume $V = 0$. Multiplying $\Psi_{2\text{D}}^*$ and integrating on both sides of equation (36) over the domain s_ϵ , in which $r = |\mathbf{r}_1 - \mathbf{r}_2| > \epsilon$, we obtain

$$E \int_{s_\epsilon} \mathbf{d}\mathbf{r}_1 \mathbf{d}\mathbf{r}_2 |\Psi_{2\text{D}}|^2 = \int_{s_\epsilon} \mathbf{d}\mathbf{r}_1 \mathbf{d}\mathbf{r}_2 \Psi_{2\text{D}}^* \sum_{i=1}^2 \left(-\frac{\hbar^2}{2M} \nabla_i^2 \right) \Psi_{2\text{D}}. \quad (37)$$

On the left-hand side (LHS) of equation (37), we already obtain

$$\int_{s_\epsilon} \mathbf{d}\mathbf{r}_1 \mathbf{d}\mathbf{r}_2 |\Psi_{2\text{D}}|^2 = 1 + \sum_{\sigma} \mathcal{I}_a^{(\sigma)} \left(\ln \frac{2b_\sigma}{\epsilon} - \gamma \right) \quad (38)$$

in appendix A. Let us concentrate on the right-hand side (RHS), which may be rewritten as

$$\text{RHS} = I_{\text{whole}}^{(2)} - I_{s_\epsilon}^{(2)}, \quad (39)$$

where

$$I_{\text{whole}}^{(2)} \equiv \int \mathbf{d}\mathbf{r}_1 \mathbf{d}\mathbf{r}_2 \Psi_{2\text{D}}^* \sum_{i=1}^2 \left(-\frac{\hbar^2}{2M} \nabla_i^2 \right) \Psi_{2\text{D}}, \quad (40)$$

$$I_{\bar{s}_\epsilon}^{(2)} \equiv \int_{\bar{s}_\epsilon} \mathbf{d}\mathbf{r}_1 \mathbf{d}\mathbf{r}_2 \Psi_{2\text{D}}^* \sum_{i=1}^2 \left(-\frac{\hbar^2}{2M} \nabla_i^2 \right) \Psi_{2\text{D}}. \quad (41)$$

Here, \bar{s}_ϵ is the complementary set of s_ϵ , in which $r < \epsilon$. If we write the two-body wave function $\Psi_{2\text{D}}$ in the momentum space, i.e.

$$\Psi_{2\text{D}} = \sum_{\mathbf{k}_1} \sum_{\mathbf{k}_2} \Phi_{2\text{D}}(\mathbf{k}_1, \mathbf{k}_2) e^{i\mathbf{k}_1 \cdot \mathbf{r}_1} e^{i\mathbf{k}_2 \cdot \mathbf{r}_2}, \quad (42)$$

where

$$\Phi_{2\text{D}}(\mathbf{k}_1, \mathbf{k}_2) \equiv \int \mathbf{d}\mathbf{r}_1 \mathbf{d}\mathbf{r}_2 \Psi_{2\text{D}} e^{-i\mathbf{k}_1 \cdot \mathbf{r}_1} e^{-i\mathbf{k}_2 \cdot \mathbf{r}_2}, \quad (43)$$

$I_{\text{whole}}^{(2)}$ becomes

$$\begin{aligned} I_{\text{whole}}^{(2)} &= \sum_{\mathbf{k}_1} \sum_{\mathbf{k}_2} \sum_{i=1}^2 \frac{\hbar^2 k_i^2}{2M} |\Phi_{2\text{D}}|^2 \\ &= \sum_{\mathbf{k}} \frac{\hbar^2 k^2}{2M} n(\mathbf{k}), \end{aligned} \quad (44)$$

where $n(\mathbf{k})$ is the total momentum distribution of two fermions. If we extend the asymptotic form of $\Psi_{2\text{D}}$ (1) to the region even inside the interaction range, the momentum distribution $n(k)$ decays like k^{-2} and k^{-4} as $k \rightarrow \infty$, respectively, as shown in equation (30), and then $I_{\text{whole}}^{(2)}$ becomes divergent. However, we will see such divergence is exactly removed by $I_{\bar{s}_\epsilon}^{(2)}$, and the RHS of equation (37), i.e. equation (39), converges.

In the following, let us focus on the integral $I_{\bar{s}_\epsilon}^{(2)}$. Inserting equations (1) into (41), and rewriting the integral in the c.m. frame of two fermions, we obtain

$$\begin{aligned} I_{\bar{s}_\epsilon}^{(2)} &= \sum_{\sigma\sigma'} \left[\int \mathbf{d}\mathbf{R} \mathcal{A}_{\sigma'}^* \mathcal{A}_\sigma \cdot \mathbf{I}_1^{(\sigma\sigma')} \right. \\ &\quad \left. + \int \mathbf{d}\mathbf{R} \mathcal{A}_{\sigma'}^* \left(-\frac{\hbar^2}{4M} \nabla_{\mathbf{R}}^2 \right) \mathcal{A}_\sigma \cdot \mathbf{I}_2^{(\sigma\sigma')} \right], \end{aligned} \quad (45)$$

where

$$\mathbf{I}_1^{(\sigma\sigma')} \equiv \int_{r < \epsilon} \mathbf{d}\mathbf{r} \psi_{\sigma'}^*(\mathbf{r}) \left(-\frac{\hbar^2}{M} \nabla_{\mathbf{r}}^2 \right) \psi_\sigma(\mathbf{r}), \quad (46)$$

$$\mathbf{I}_2^{(\sigma\sigma')} \equiv \int_{r < \epsilon} \mathbf{d}\mathbf{r} \psi_{\sigma'}^*(\mathbf{r}) \psi_\sigma(\mathbf{r}), \quad (47)$$

and we should note that the variable \mathbf{X} in the function \mathcal{A} drops out automatically for a two-body system. Let us calculate $\mathbf{I}_1^{(\sigma\sigma')}$ first, and keep in mind that $\psi_\sigma(\mathbf{r})$ takes the form of equation (2), which is the linear combination of $J_1(qr) \Omega_1^{(\sigma)}(\varphi)$ and $N_1(qr) \Omega_1^{(\sigma)}(\varphi)$, respectively, the regular and irregular solutions of the p -wave Schrödinger equation. Therefore, we have

$$\nabla_{\mathbf{r}}^2 \left[J_1(qr) \Omega_1^{(\sigma)}(\varphi) \right] = -q^2 J_1(qr) \Omega_1^{(\sigma)}(\varphi), \quad (48)$$

and then we easily find

$$\begin{aligned} \int_{r<\epsilon} d\mathbf{r} \psi_{\sigma'}^*(\mathbf{r}) \left(-\frac{\hbar^2}{M} \nabla_{\mathbf{r}}^2 \right) \left[J_1(qr) \Omega_1^{(\sigma)}(\varphi) \right] \\ = \delta_{\sigma\sigma'} \left(\frac{\hbar^2 q}{4M} \right) \left[(\epsilon q)^2 + O(\epsilon q)^4 \right], \end{aligned} \quad (49)$$

which vanishes in the low-energy limit, i.e. $\epsilon q \rightarrow 0$, and it yields

$$\mathbf{I}_1^{(\sigma\sigma')} = \frac{\pi \hbar^2 q}{2M} \int_{r<\epsilon} d\mathbf{r} \psi_{\sigma'}^*(\mathbf{r}) \nabla_{\mathbf{r}}^2 \left[N_1(qr) \Omega_1^{(\sigma)}(\varphi) \right]. \quad (50)$$

As to the irregular solution $N_1(qr) \Omega_1^{(\sigma)}(\varphi)$, we have (see appendix C)

$$\nabla_{\mathbf{r}}^2 \left[N_1(qr) \Omega_1^{(\sigma)}(\varphi) \right] = \left[\frac{4\delta(r)}{\pi q r^2} - q^2 N_1(qr) \right] \Omega_1^{(\sigma)}(\varphi), \quad (51)$$

and then

$$\begin{aligned} \mathbf{I}_1^{(\sigma\sigma')} = \delta_{\sigma\sigma'} \frac{\hbar^2}{M} \left(\frac{\pi}{2} q^2 \cot \delta_{\sigma} \right) \\ - \frac{\pi^2 \hbar^2 q^2}{4M} \int_{r<\epsilon} d\mathbf{r} N_1(qr) \Omega_1^{(\sigma')*}(\varphi) \nabla_{\mathbf{r}}^2 \left[N_1(qr) \Omega_1^{(\sigma)}(\varphi) \right]. \end{aligned} \quad (52)$$

Apparently, the last term of equation (52) is divergent, since the Bessel function $N_1(qr)$ behaves as

$$N_1(qr) = -\frac{2}{\pi q r} + \frac{q r}{\pi} \left(\ln \frac{q r}{2} + \gamma - \frac{1}{2} \right) + O(qr)^3 \quad (53)$$

at $qr \sim 0$. The crucial point of the energy theorem is to discuss such divergence alternatively in the momentum space. After straightforward algebra, we show in the appendix D that

$$\begin{aligned} \frac{\pi^2 \hbar^2 q^2}{4M} \int_{r<\epsilon} d\mathbf{r} N_1(qr) \Omega_1^{(\sigma')*}(\varphi) \nabla_{\mathbf{r}}^2 \left[N_1(qr) \Omega_1^{(\sigma)}(\varphi) \right] \\ = \delta_{\sigma\sigma'} \lim_{\Lambda \rightarrow \infty} \left[-\frac{\hbar^2 \Lambda^2}{2M} - \frac{\hbar^2 q^2}{M} \ln \frac{\Lambda}{q} - \frac{\hbar^2 q^2}{M} \left(\gamma + \ln \frac{\epsilon \Lambda}{2} \right) - \int_{\Lambda}^{\infty} \frac{k dk}{(2\pi)^2} \frac{\hbar^2 k^2}{2M} \left(\frac{8\pi^2}{k^2} + \frac{16\pi^2 q^2}{k^4} \right) \right], \end{aligned} \quad (54)$$

and then it yields

$$\begin{aligned} \mathbf{I}_1^{(\sigma\sigma')} = \delta_{\sigma\sigma'} \lim_{\Lambda \rightarrow \infty} \left[\frac{\hbar^2}{2M} \left(\Lambda^2 - \frac{\pi}{a_{\sigma}} \right) + \frac{\hbar^2 q^2}{M} \ln(\Lambda b_{\sigma}) + \frac{\hbar^2 q^2}{M} \left(\gamma + \ln \frac{\epsilon \Lambda}{2} \right) \right. \\ \left. + \int_{\Lambda}^{\infty} \frac{k dk}{(2\pi)^2} \frac{\hbar^2 k^2}{2M} \left(\frac{8\pi^2}{k^2} + \frac{16\pi^2 q^2}{k^4} \right) \right], \end{aligned} \quad (55)$$

where we have used the effective expansion (7). In the expression of equation (55), we exactly separate the divergent part of $\mathbf{I}_1^{(\sigma\sigma')}$ in the momentum space as appearing in the last integral.

As to the integral $\mathbf{I}_2^{(\sigma\sigma')}$, i.e. equation (47), we easily find it is also divergent, since the wave function $\psi_\sigma(\mathbf{r})$ behaves as r^{-1} at $r \sim 0$. Such divergence can also be separated in the momentum space (see appendix D), and yields

$$\mathbf{I}_2^{(\sigma\sigma')} = \delta_{\sigma\sigma'} \lim_{\Lambda \rightarrow \infty} \left[\gamma + \ln \frac{\epsilon\Lambda}{2} + 4\pi^2 \int_{\Lambda}^{\infty} \frac{kdk}{(2\pi)^2} \frac{1}{k^2} \right]. \quad (56)$$

Combining equations (45), (55) and (56), we obtain

$$I_{\bar{\epsilon}}^{(2)} = \lim_{\Lambda \rightarrow \infty} \left\{ \sum_{\sigma} \frac{\hbar^2 c_a^{(\sigma)}}{16\pi^2 M} \left(\Lambda^2 - \frac{\pi}{a_{\sigma}} \right) + \frac{\hbar^2}{8\pi^2 M} \sum_{\sigma} \left[\frac{c_b^{(\sigma)}}{2} \ln(\Lambda b_{\sigma}) + \left(\frac{c_b^{(\sigma)}}{2} + Q_{\text{cm}}^{(\sigma)} \right) \left(\ln \frac{\epsilon\Lambda}{2} + \gamma \right) \right] + \int_{\Lambda}^{\infty} \frac{kdk}{(2\pi)^2} \frac{\hbar^2 k^2}{2M} \left[\frac{\sum_{\sigma} c_a^{(\sigma)}}{k^2} + \frac{\sum_{\sigma} (c_b^{(\sigma)} + Q_{\text{cm}}^{(\sigma)})}{k^4} \right] \right\}, \quad (57)$$

where we have defined the corresponding two-body quantities $c_a^{(\sigma)} \equiv 8\pi^2 \mathcal{I}_a^{(\sigma)}$, $c_b^{(\sigma)} \equiv -32\pi^2 \mathcal{I}_b^{(\sigma)}$, and $Q_{\text{cm}}^{(\sigma)} \equiv 2\pi^2 \int d\mathbf{R} (\nabla_{\mathbf{R}} \mathcal{A}_{\sigma}^* \cdot \nabla_{\mathbf{R}} \mathcal{A}_{\sigma})$. We can see that the divergent integral of $I_{\bar{\epsilon}}^{(2)}$ exactly compensates that of $I_{\text{whole}}^{(2)}$, and then the RHS of equation (37), i.e. $I_{\text{whole}}^{(2)} - I_{\bar{\epsilon}}^{(2)}$ converges.

The above procedure can easily be generalized to the many-body system of N spin-polarized fermions. The divergence of the corresponding integral $I_{\text{whole}}^{(N)}$ arises when any two of fermions get close. Since there are totally $\mathcal{N} = N(N-1)/2$ ways to pair atoms, we obtain $I_{\bar{\mathcal{S}}_{\epsilon}}^{(N)} = \mathcal{N} I_{\bar{\epsilon}}^{(2)}$, where the domain $\bar{\mathcal{S}}_{\epsilon}$ is the set of all configurations $(\mathbf{r}_i, \mathbf{r}_j)$, in which $|\mathbf{r}_i - \mathbf{r}_j| < \epsilon$. Finally, after redefining the constant \mathcal{N} into the contacts, the energy theorem for a many-body system can be rearranged as

$$E \left[1 + \sum_{\sigma} \frac{C_a^{(\sigma)}}{8\pi^2} \left(\ln \frac{2b_{\sigma}}{\epsilon} - \gamma \right) \right] = \lim_{\Lambda \rightarrow \infty} \left\{ \sum_{|\mathbf{k}| < \Lambda} \frac{\hbar^2 k^2}{2M} n(\mathbf{k}) - \frac{\hbar^2}{16\pi^2 M} \sum_{\sigma} C_a^{(\sigma)} \left(\Lambda^2 - \frac{\pi}{a_{\sigma}} \right) - \frac{\hbar^2}{8\pi^2 M} \sum_{\sigma} \left[\frac{C_b^{(\sigma)}}{2} \ln(\Lambda b_{\sigma}) + \left(\frac{C_b^{(\sigma)}}{2} + Q_{\text{cm}}^{(\sigma)} \right) \left(\ln \frac{\epsilon\Lambda}{2} + \gamma \right) \right] \right\}, \quad (58)$$

where γ is the Euler's constant. Here, we should note that unlike the s -wave case, the range ϵ of the p -wave interaction appears in the energy theorem. This feature results from the non-normalizability of the higher-partial wave functions, and the short-range physics becomes important, which has already been pointed out in [31] for the three-dimensional (3D) systems. The parameter Λ in the expression of the limit characterizes the cut-off of the momentum k , and it screens off the details of interatomic interactions from the outside. Therefore, the order of Λ should physically be larger than the inverse of the interaction range ϵ .

5. The high-frequency tail of the RF spectroscopy

In the RF experiments, the fermions can be driven from the initially occupied spin state $|g\rangle$ to an *empty* spin state $|e\rangle$, when the external RF field is tuned near the transition frequency between the states $|g\rangle$ and $|e\rangle$. The universal scaling behavior at high frequency of the RF response of the system is governed by contacts [6, 9, 30, 33, 37]. In this section, we are going to show how the contacts defined by the adiabatic relations characterize such high-frequency

scalings of the RF spectroscopy of a spin-polarized Fermi gas in two dimensions. Let us again start from a two-body picture, and consider two fermions in the same spin state, which may simplify the presentation as much as possible. The RF field is described in the momentum space by

$$\mathcal{H}_{\text{rf}} = \gamma_{\text{rf}} \sum_{\mathbf{k}} \left(e^{-i\omega t} c_{e\mathbf{k}}^\dagger c_{\mathbf{k}} + e^{i\omega t} c_{\mathbf{k}}^\dagger c_{e\mathbf{k}} \right), \quad (59)$$

where γ_{rf} is the strength of the RF drive, ω is the RF, and $c_{e\mathbf{k}}^\dagger$ and $c_{\mathbf{k}}^\dagger$ are respectively the creation operators for fermions with the momentum \mathbf{k} in the spin states $|e\rangle$ and $|g\rangle$. The initial two-body state before the RF transition can be written as

$$|\Psi_i\rangle = \frac{1}{\sqrt{2}} \sum_{\mathbf{k}_1, \mathbf{k}_2} \tilde{\Psi}_{2\text{D}}(\mathbf{k}_1, \mathbf{k}_2) c_{\mathbf{k}_1}^\dagger c_{\mathbf{k}_2}^\dagger |0\rangle, \quad (60)$$

where $\tilde{\Psi}_{2\text{D}}(\mathbf{k}_1, \mathbf{k}_2)$ is the Fourier transform of the two-body wave function (1) (the variable \mathbf{X} drops out in the two-body picture), i.e.

$$\tilde{\Psi}_{2\text{D}}(\mathbf{k}_1, \mathbf{k}_2) = \int d\mathbf{r}_1 d\mathbf{r}_2 \Psi_{2\text{D}}(\mathbf{r}_1, \mathbf{r}_2) e^{-i\mathbf{k}_1 \cdot \mathbf{r}_1} e^{-i\mathbf{k}_2 \cdot \mathbf{r}_2}. \quad (61)$$

Acting equations (59) onto (60), we easily obtain the final state after the RF transition,

$$|\Psi_f\rangle = \frac{\gamma_{\text{rf}} e^{-i\omega t}}{\sqrt{2}} \sum_{\mathbf{k}_1, \mathbf{k}_2} \tilde{\Psi}_{2\text{D}}(\mathbf{k}_1, \mathbf{k}_2) \left(c_{e\mathbf{k}_1}^\dagger c_{\mathbf{k}_2}^\dagger - c_{e\mathbf{k}_2}^\dagger c_{\mathbf{k}_1}^\dagger \right) |0\rangle. \quad (62)$$

The physical meaning of equation (62) is quite apparent: after the RF transition, there are two possible final states that one of the two fermions is driven from the initial spin state $|g\rangle$ to the final spin state $|e\rangle$ with either momentum \mathbf{k}_1 or \mathbf{k}_2 , and the probabilities are both $\gamma_{\text{rf}}^2 |\tilde{\Psi}_{2\text{D}}(\mathbf{k}_1, \mathbf{k}_2)|^2 / 2$. According to the Fermi's golden rule [30], and taking these two final states into account, the two-body RF transition rate takes the form

$$\Gamma_2(\omega) = \frac{\pi \gamma_{\text{rf}}^2}{\hbar} \sum_{\mathbf{k}_1, \mathbf{k}_2} \left| \tilde{\Psi}_{2\text{D}}(\mathbf{k}_1, \mathbf{k}_2) \right|^2 \delta(\hbar\omega - \Delta E), \quad (63)$$

where ΔE is the energy difference between the final and initial states. If the final spin state $|e\rangle$ has an ignorable interaction with the initial spin state $|g\rangle$, the final-state energy becomes

$$E_f = \frac{\hbar^2 K^2}{4M} + \frac{\hbar^2 k^2}{M} + \hbar\omega_e + \hbar\omega_g, \quad (64)$$

where $\mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2$, $\mathbf{k} = (\mathbf{k}_1 - \mathbf{k}_2) / 2$, and ω_e and ω_g are the bare hyperfine frequencies of the final and initial spin states, respectively. The energy of the initial state with two fermions in the spin state $|g\rangle$ is

$$E_i = \frac{\hbar^2 K^2}{4M} + \frac{\hbar^2 q^2}{M} + 2\hbar\omega_g, \quad (65)$$

and $\hbar^2 q^2 / M$ is the relative energy of two fermions in the spin state $|g\rangle$. Therefore, we have

$$\Delta E \approx \frac{\hbar^2 k^2}{M} + \hbar(\omega_e - \omega_g) - \frac{\hbar^2 q^2}{M}, \quad (66)$$

then we obtain

$$\Gamma_2(\omega) = \frac{\pi\gamma_{\text{rf}}^2}{\hbar} \sum_{\mathbf{k}_1, \mathbf{k}_2} \left| \tilde{\Psi}_{2\text{D}}(\mathbf{k}_1, \mathbf{k}_2) \right|^2 \delta \left(\hbar\omega + \frac{\hbar^2(q^2 - k^2)}{M} \right), \quad (67)$$

where we set the bare hyperfine splitting $\omega_e - \omega_g = 0$ without loss of generality. Furthermore, if inserting equations (61) into (67), we may rewrite the RF transition rate as

$$\Gamma_2(\omega) = \frac{\pi\gamma_{\text{rf}}^2}{\hbar} \sum_{\mathbf{k}} n'(\mathbf{k}) \delta \left(\hbar\omega + \frac{\hbar^2(q^2 - k^2)}{M} \right), \quad (68)$$

where

$$n'(\mathbf{k}) \equiv \int d\mathbf{R} \left| \int d\mathbf{r} \Psi_{2\text{D}} e^{-i\mathbf{k}\cdot\mathbf{r}} \right|^2, \quad (69)$$

and recall $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$ and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. Comparing equation (69) with the definition of the single-particle momentum distribution, i.e. equation (16), or specifically, for a two-body system

$$n_1(\mathbf{k}) = \int d\mathbf{r}_2 \left| \int d\mathbf{r}_1 \tilde{\Psi}_{2\text{D}} e^{-i\mathbf{k}\cdot\mathbf{r}_1} \right|^2, \quad (70)$$

we find $n'(\mathbf{k})$ is not exactly the single-particle momentum distribution of the system: $n'(\mathbf{k})$ should have the same leading-order behavior as that of $n_1(\mathbf{k})$ at large \mathbf{k} , but different sub-leading-order behavior, in which the c.m. contribution is excluded. As we can see from equation (68), the high-frequency behavior of the RF transition rate is determined by $n'(\mathbf{k})$ at large \mathbf{k} but still smaller than ϵ^{-1} , due to the delta function. Here, we should carefully deal with the relative energy $\hbar^2 q^2/M$ in the low-energy limit, since the subleading-order behavior becomes important. Simply using the Fourier transform of the relative wave function of two fermions at small \mathbf{r} , we easily obtain the form of $n'(\mathbf{k})$ at large \mathbf{k} , and then the two-body RF transition rate $\Gamma_2(\omega)$ becomes

$$\Gamma_2(\omega) \approx \frac{M\gamma_{\text{rf}}^2}{16\pi\hbar^3} \left[\frac{\sum_{\sigma} c_a^{(\sigma)}}{M\omega/\hbar} + \frac{\sum_{\sigma} c_b^{(\sigma)}}{2(M\omega/\hbar)^2} \right] \quad (71)$$

at large ω but smaller than $\hbar/M\epsilon^2$, and again $c_a^{(\sigma)}, c_b^{(\sigma)}$ are the two-body contacts.

For the many-body systems, all possible \mathcal{N} pairs may contribute to the high-frequency tail of the RF spectroscopy, when the two fermions in them get close, while all the other fermions are far away. Therefore, we can follow the above two-body route, and easily obtain the asymptotic behavior of the RF response of the many-body system at large ω , after redefining the constant \mathcal{N} into the contacts, i.e.

$$\Gamma(\omega) \approx \frac{M\gamma_{\text{rf}}^2}{16\pi\hbar^3} \left[\frac{\sum_{\sigma} \mathcal{C}_a^{(\sigma)}}{M\omega/\hbar} + \frac{\sum_{\sigma} \mathcal{C}_b^{(\sigma)}}{2(M\omega/\hbar)^2} \right], \quad (72)$$

where $\mathcal{C}_a^{(\sigma)}$ and $\mathcal{C}_b^{(\sigma)}$ are corresponding many-body contacts, and $\Gamma(\omega)$ should obey the sum rule $\int d\omega \Gamma(\omega) = \pi\gamma_{\text{rf}}^2 N/\hbar^2$ [37].

6. Pair correlation function at short distances

The pair correlation function $g_2(\mathbf{s}, \mathbf{t})$ gives the probability of finding two fermions at positions \mathbf{s} and \mathbf{t} simultaneously, i.e. $g_2(\mathbf{s}, \mathbf{t}) \equiv \langle \hat{\rho}(\mathbf{s}) \hat{\rho}(\mathbf{t}) \rangle$, where $\hat{\rho}(\mathbf{s}) = \sum_i \delta(\mathbf{s} - \mathbf{r}_i)$ is the density operator at the position \mathbf{s} . For a pure many-body state $|\Psi_{2D}\rangle$ of N fermions, we have

$$\begin{aligned} g_2(\mathbf{s}, \mathbf{t}) &= \int d\mathbf{r}_1 d\mathbf{r}_2 \cdots d\mathbf{r}_N \langle \Psi_{2D} | \hat{\rho}(\mathbf{s}) \hat{\rho}(\mathbf{t}) | \Psi_{2D} \rangle \\ &= N(N-1) \int d\mathbf{X} |\Psi_{2D}(\mathbf{X}, \mathbf{R}, \mathbf{r})|^2, \end{aligned} \quad (73)$$

where $\mathbf{R} = (\mathbf{s} + \mathbf{t})/2$, $\mathbf{r} = \mathbf{s} - \mathbf{t}$, and \mathbf{X} denotes all the degrees of freedom of the fermions except the ones at \mathbf{s} and \mathbf{t} . Further more, we may also integrate over the c.m. coordinate \mathbf{R} , and define the spatially integrated pair correlation function as

$$G_2(\mathbf{r}) \equiv \int d\mathbf{R} g_2\left(\mathbf{R} + \frac{\mathbf{r}}{2}, \mathbf{R} - \frac{\mathbf{r}}{2}\right). \quad (74)$$

Using the asymptotic form the many-body wave function at short distance, i.e. equation (1), we easily obtain

$$\begin{aligned} G_2(\mathbf{r}) &\approx N(N-1) \sum_{\sigma\sigma'} \int d\mathbf{X} d\mathbf{R} \mathcal{A}_{\sigma'}^* \mathcal{A}_{\sigma} \\ &\left[\frac{1}{r^2} - \frac{q^2}{2} \left(\ln \frac{r}{2b_{\sigma'}} + \ln \frac{r}{2b_{\sigma}} \right) \right] \Omega_{\sigma'}^*(\varphi) \Omega_{\sigma}(\varphi). \end{aligned} \quad (75)$$

If we are only interested in the dependence of $G_2(\mathbf{r})$ on $r = |\mathbf{r}|$, we may integrate $G_2(\mathbf{r})$ over the direction of \mathbf{r} , and obtain

$$\begin{aligned} G_2(r) &\approx N(N-1) \sum_{\sigma} \left(\frac{\mathcal{I}_a^{(\sigma)}}{r^2} + 2\mathcal{I}_b^{(\sigma)} \ln \frac{r}{2b_{\sigma}} \right) \\ &= \frac{1}{4\pi^2} \sum_{\sigma} \left(\frac{\mathcal{C}_a^{(\sigma)}}{r^2} - \frac{\mathcal{C}_b^{(\sigma)}}{2} \ln \frac{r}{2b_{\sigma}} \right). \end{aligned} \quad (76)$$

We can see that the short-distance behavior of the pair correlation function of a spin-polarized Fermi gas is also completely captured by the p -wave contacts $\mathcal{C}_a^{(\sigma)}$ and $\mathcal{C}_b^{(\sigma)}$.

7. Generalized virial theorem and pressure relation

Let us consider a spin-polarized Fermi gas trapped in the harmonic potential V , then the Helmholtz free energy F should be the function of the temperature T , the trap frequency ω , the atom number N , and the interatomic p -wave interaction strength characterized by the 2D scattering area a_{σ} as well as the effective range b_{σ} , i.e. $F(T, \omega, a_{\sigma}, b_{\sigma}, N)$. The generalized virial theorem can be obtained according to the dimensional analysis [7, 8, 38]. Using $\hbar\omega$ as the unit of the energy, the Helmholtz free energy may be written as

$$F(T, \omega, a_{\sigma}, b_{\sigma}, N) = \hbar\omega f\left(\frac{k_B T}{\hbar\omega}, \frac{\hbar^2/Ma_{\sigma}}{\hbar\omega}, \frac{\hbar^2/Mb_{\sigma}^2}{\hbar\omega}, N\right), \quad (77)$$

where the function f is just a dimensionless function, and k_B is the Boltzmann constant. Then the free energy F should have the following scaling property,

$$F(\lambda T, \lambda \omega, \lambda^{-1} a_\sigma, \lambda^{-1/2} b_\sigma, N) = \lambda F(T, \omega, a_\sigma, b_\sigma, N). \quad (78)$$

Taking the derivative with respect to λ on both sides of equation (78), and then setting $\lambda = 1$, we obtain

$$\left(T \frac{\partial}{\partial T} + \omega \frac{\partial}{\partial \omega} - a_\sigma \frac{\partial}{\partial a_\sigma} - \frac{b_\sigma}{2} \frac{\partial}{\partial b_\sigma} \right) F = F. \quad (79)$$

Since the Helmholtz free energy is just the Legendre transform of the energy, its partial derivatives at constant T with respect to ω , a_σ , and b_σ are equal to those of the energy at the associated value of the entropy S . Combining the adiabatic relations (34) and (35), and $dF = dE - SdT$, we easily obtain

$$E = 2 \langle V \rangle - \frac{\hbar^2 \mathcal{C}_a^{(\sigma)}}{16\pi M a_\sigma} - \frac{\hbar^2 \mathcal{C}_b^{(\sigma)}}{32\pi^2 M}. \quad (80)$$

The pressure relation can be derived following the similar route. Let us consider the free energy density \mathcal{F} , which has the dimension of (energy)² up to the factors \hbar and M . Assuming κ is an arbitrary quantity with dimension of (energy)¹, the free energy density can be written as

$$\mathcal{F}(T, a_\sigma, b_\sigma, n) = \frac{M\kappa^2}{\hbar^2} f\left(\frac{k_B T}{\kappa}, \frac{\hbar^2/Ma_\sigma}{\kappa}, \frac{\hbar^2/Mb_\sigma^2}{\kappa}, \frac{\hbar^2 n/M}{\kappa}\right), \quad (81)$$

where n is the atom density. Then we have

$$\mathcal{F}(\lambda T, \lambda^{-1} a_\sigma, \lambda^{-1/2} b_\sigma, \lambda n) = \lambda^2 \mathcal{F}(T, a_\sigma, b_\sigma, n), \quad (82)$$

which similarly yields

$$\left(T \frac{\partial}{\partial T} - a_\sigma \frac{\partial}{\partial a_\sigma} - \frac{b_\sigma}{2} \frac{\partial}{\partial b_\sigma} + n \frac{\partial}{\partial n} \right) \mathcal{F} = 2\mathcal{F}. \quad (83)$$

Combining $P = -\mathcal{F} + n\mu$, where μ is the chemical potential, and the adiabatic relations, we finally obtain the pressure relation

$$P = \varepsilon + \frac{\hbar^2 \mathcal{C}_a^{(\sigma)}}{16\pi M a_\sigma} + \frac{\hbar^2 \mathcal{C}_b^{(\sigma)}}{32\pi^2 M}, \quad (84)$$

where ε is the energy density of the system.

8. Conclusions

To conclude, we have systematically studied the full set of universal relations of a 2D spin-polarized Fermi gas with p -wave interactions. If the p -wave contacts are defined according to the adiabatic relations, we find that the universal relations of the system, such as the high-frequency tail of the RF response, short-distance behavior of the pair correlation function, generalized virial theorem, and pressure relation are fully captured by the contacts we define. As we anticipate, an extra term resulted from the center-of-mass motions of the pairs appears in the subleading tail (k^{-4}) of the large momentum distribution besides the contact related to the effective range, similar to what happens in a 3D p -wave Fermi gas. Furthermore, such an extra term results in an additional divergence for the energy theorem, which should carefully

be handled with. We show that all the divergences of the kinetic energy are exactly compensated by the interatomic interaction energy, and the total internal energy of the system converges. Our results could easily be generalized for higher-partial-wave scatterings. The predicted universal relations could readily be confirmed in current cold-atom experiments with spin-polarized Fermi gases of ^{40}K and ^6Li atoms.

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Appendix A. The normalization of the wave function

In this appendix, we are going to discuss the normalization of the many-body wave function $\Psi_{2\text{D}}$ in two dimension, and calculate $\int_{S_\epsilon} \prod_{i=1}^N \mathbf{d}\mathbf{r}_i |\Psi_{2\text{D}}|^2$. The similar normalization has been discussed for a 3D system (see appendix A of [32]), and such normalization is related to the probability of finding two fermions inside the interaction range ϵ . Let us consider N spin-polarized fermions, and when any two of them interact with each other, for example, fermions i and j , all the others are far away. For two different relative energies of the pair (i, j) , i.e. \mathcal{E} and \mathcal{E}' , due to the orthogonality of the wave function, we have $\int \mathbf{d}\mathbf{X}\mathbf{d}\mathbf{R}\mathbf{d}\mathbf{r} \Psi_{2\text{D}}'^* \Psi_{2\text{D}} = 0$. Then from the Schrödinger equations satisfied by $\Psi_{2\text{D}}'$ and $\Psi_{2\text{D}}$, we find the probability of finding the pair (i, j) inside the interaction range ϵ should be

$$\begin{aligned}
\int_{r<\epsilon} \mathbf{d}\mathbf{X}\mathbf{d}\mathbf{R}\mathbf{d}\mathbf{r} |\Psi_{2\text{D}}|^2 &= - \lim_{\mathcal{E}' \rightarrow \mathcal{E}} \int_{r>\epsilon} \mathbf{d}\mathbf{X}\mathbf{d}\mathbf{R}\mathbf{d}\mathbf{r} \Psi_{2\text{D}}'^* \Psi_{2\text{D}} \\
&= - \lim_{\mathcal{E}' \rightarrow \mathcal{E}} \frac{\hbar^2/M}{\mathcal{E} - \mathcal{E}'} \int_{r=\epsilon} \mathbf{d}\mathbf{X}\mathbf{d}\mathbf{R}\mathbf{d}\mathbf{r} \left(\Psi_{2\text{D}}'^* \frac{\partial}{\partial r} \Psi_{2\text{D}} - \Psi_{2\text{D}} \frac{\partial}{\partial r} \Psi_{2\text{D}}'^* \right) \\
&= - \lim_{\mathcal{E}' \rightarrow \mathcal{E}} \sum_{\sigma\sigma'} \int \mathbf{d}\mathbf{X}\mathbf{d}\mathbf{R} \mathcal{A}_{\sigma\sigma'}^* \mathcal{A}_\sigma \cdot \frac{\hbar^2/M}{\mathcal{E} - \mathcal{E}'} \int_0^{2\pi} \epsilon d\varphi \left(\psi_{\sigma'}'^* \frac{\partial}{\partial r} \psi_\sigma - \psi_\sigma \frac{\partial}{\partial r} \psi_{\sigma'}'^* \right) \\
&= - \sum_{\sigma} \mathcal{I}_a^{(\sigma)} \left[- \left(\gamma + \ln \frac{\epsilon}{2} \right) + \frac{\partial}{\partial q^2} \left(\frac{\pi}{2} q^2 \cot \delta_\sigma - q^2 \ln q \right) \right] \\
&= - \sum_{\sigma} \mathcal{I}_a^{(\sigma)} \left(\ln \frac{2b_\sigma}{\epsilon} - \gamma \right), \tag{A.1}
\end{aligned}$$

where we have used the effective-range expansion of the scattering phase shift (7), $q^2 = M\mathcal{E}/\hbar^2$, and γ is the Euler's constant. We can see that the bound for the effective range b_σ exists, i.e. $b_\sigma < \epsilon e^\gamma/2$, in order to guarantee the positive probability of finding two atoms inside the interaction range. This is an alternative expression of the Wigner's bound on the effective range for the p -wave interaction in two dimensions [32, 39–41]. Then the total probability of finding any pair of fermions inside the interaction range is

$$\int_{S_\epsilon} \prod_{i=1}^N \mathbf{d}\mathbf{r}_i |\Psi_{2\text{D}}|^2 = \frac{N(N-1)}{2} \int_{r<\epsilon} \mathbf{d}\mathbf{X}\mathbf{d}\mathbf{R}\mathbf{d}\mathbf{r} |\Psi_{2\text{D}}|^2 = -N \sum_{\sigma} \mathcal{I}_a^{(\sigma)} \left(\ln \frac{2b_\sigma}{\epsilon} - \gamma \right), \tag{A.2}$$

where $\mathcal{I}_a^{(\sigma)}$ is defined in equation (10), and $\bar{\mathcal{S}}_\epsilon$ is the set of all configurations that there is only one pair inside the interaction range. Consequently, we obtain

$$\int_{\bar{\mathcal{S}}_\epsilon} \prod_{i=1}^N d\mathbf{r}_i |\Psi_{2D}|^2 = 1 + \mathcal{N} \sum_{\sigma} \mathcal{I}_a^{(\sigma)} \left(\ln \frac{2b_\sigma}{\epsilon} - \gamma \right). \quad (\text{A.3})$$

Appendix B. Derivation details of the momentum distribution

In this part of the appendix, we present the calculation details in the derivation from equations (29) to (30). The integral over the direction of \mathbf{k} for the leading-order term, i.e. $\sim k^{-2}$, can easily be obtained, and the coefficient takes the simple form of $\sum_{\sigma} 8\pi^2 \mathcal{N} \mathcal{I}_a^{(\sigma)}$, which then we define as $\sum_{\sigma} \mathcal{C}_a^{(\sigma)}$. For the k^{-3} -order term, we find

$$\begin{aligned} \tau_3 &\equiv \mathbf{Im} \mathcal{N} \int d\mathbf{X} d\mathbf{R} \sum_{\sigma\sigma'} \mathcal{A}_\sigma e^{i\sigma\varphi_{\mathbf{k}}} \alpha_{\sigma'}^* \cdot \frac{4\pi}{k^3} \\ &= -4\pi \mathcal{N} \mathbf{Im} \int d\mathbf{X} d\mathbf{R} \sum_{\sigma\sigma'} \mathcal{A}_\sigma \left[\left(\nabla_{\mathbf{R}} \mathcal{A}_{\sigma'}^* \cdot \hat{\mathbf{k}} \right) + i\sigma' \left(\nabla_{\mathbf{R}} \mathcal{A}_{\sigma'}^* \cdot \hat{\varphi}_{\mathbf{k}} \right) \right] \cdot \frac{e^{i(\sigma-\sigma')\varphi_{\mathbf{k}}}}{k^3}, \end{aligned} \quad (\text{B.1})$$

where $\hat{\mathbf{k}}$, $\hat{\varphi}_{\mathbf{k}}$ are respectively the unit vectors of the radial and azimuthal directions of \mathbf{k} . Obviously, τ_3 is simply the linear combination of $e^{i(\sigma-\sigma')\varphi_{\mathbf{k}}} \sin \varphi_{\mathbf{k}}$ and $e^{i(\sigma-\sigma')\varphi_{\mathbf{k}}} \cos \varphi_{\mathbf{k}}$, which automatically vanishes if integrating over $\varphi_{\mathbf{k}}$.

Let us look at the k^{-4} -order term, which includes two terms. The first term becomes $-32\pi^2 \sum_{\sigma} \mathcal{N} \mathcal{I}_b^{(\sigma)} / k^4$ if integrating over $\varphi_{\mathbf{k}}$, and then we define as $\sum_{\sigma} \mathcal{C}_b^{(\sigma)} / k^4$. As to the second term, we rewrite it as

$$\begin{aligned} \chi &= \pi \mathcal{N} \int d\mathbf{X} d\mathbf{R} \sum_{\sigma\sigma'} [\nabla_{\mathbf{R}} \mathcal{A}_{\sigma'}^* \cdot \nabla_{\mathbf{k}} f_{\sigma'}^*] [\nabla_{\mathbf{R}} \mathcal{A}_\sigma \cdot \nabla_{\mathbf{k}} f_\sigma] \\ &\quad - \frac{\pi}{2} \mathcal{N} \int d\mathbf{X} d\mathbf{R} \sum_{\sigma\sigma'} \{ \mathcal{A}_{\sigma'}^* f_{\sigma'}^* (\nabla_{\mathbf{R}} \cdot \nabla_{\mathbf{k}}) [\nabla_{\mathbf{R}} \mathcal{A}_\sigma \cdot \nabla_{\mathbf{k}} f_\sigma] + \mathcal{A}_\sigma f_\sigma (\nabla_{\mathbf{R}} \cdot \nabla_{\mathbf{k}}) [\nabla_{\mathbf{R}} \mathcal{A}_{\sigma'}^* \cdot \nabla_{\mathbf{k}} f_{\sigma'}^*] \}, \end{aligned} \quad (\text{B.2})$$

where we have defined $f_\sigma(\mathbf{k}) \equiv e^{i\sigma\varphi_{\mathbf{k}}} / k$. Since the function \mathcal{A}_σ is regular and should either decay to zero at infinity or satisfy a periodic boundary condition in a box [4], after partially integrating, χ becomes

$$\begin{aligned} \chi &= \pi \mathcal{N} \int d\mathbf{X} d\mathbf{R} \sum_{\sigma\sigma'} [\nabla_{\mathbf{R}} \mathcal{A}_{\sigma'}^* \cdot \nabla_{\mathbf{k}} f_{\sigma'}^*] [\nabla_{\mathbf{R}} \mathcal{A}_\sigma \cdot \nabla_{\mathbf{k}} f_\sigma] \\ &\quad + \frac{\pi}{2} \mathcal{N} \int d\mathbf{X} d\mathbf{R} \sum_{\sigma\sigma'} \{ f_{\sigma'}^* (\nabla_{\mathbf{R}} \mathcal{A}_{\sigma'}^* \cdot \nabla_{\mathbf{k}}) [\nabla_{\mathbf{R}} \mathcal{A}_\sigma \cdot \nabla_{\mathbf{k}} f_\sigma] + f_\sigma (\nabla_{\mathbf{R}} \mathcal{A}_\sigma \cdot \nabla_{\mathbf{k}}) [\nabla_{\mathbf{R}} \mathcal{A}_{\sigma'}^* \cdot \nabla_{\mathbf{k}} f_{\sigma'}^*] \} \\ &= \frac{\pi}{2} \mathcal{N} \int d\mathbf{X} d\mathbf{R} \sum_{\sigma\sigma'} \sum_{ij} \frac{\partial \mathcal{A}_{\sigma'}^*}{\partial R_i} \frac{\partial \mathcal{A}_\sigma}{\partial R_j} \left(2 \frac{\partial f_{\sigma'}^*}{\partial k_i} \frac{\partial f_\sigma}{\partial k_j} + f_{\sigma'}^* \frac{\partial^2 f_\sigma}{\partial k_i \partial k_j} + f_\sigma \frac{\partial^2 f_{\sigma'}^*}{\partial k_j \partial k_i} \right) \\ &= \frac{\pi}{2} \mathcal{N} \int d\mathbf{X} d\mathbf{R} \sum_{\sigma\sigma'} \sum_{ij} \frac{\partial \mathcal{A}_{\sigma'}^*}{\partial R_i} \frac{\partial \mathcal{A}_\sigma}{\partial R_j} \left[\frac{\partial}{\partial k_i} \left(f_{\sigma'}^* \frac{\partial f_\sigma}{\partial k_j} \right) + \frac{\partial}{\partial k_j} \left(f_\sigma \frac{\partial f_{\sigma'}^*}{\partial k_i} \right) \right], \end{aligned} \quad (\text{B.3})$$

where the indices i, j denote $\{x, y, z\}$. Inserting

$$f_\sigma(\mathbf{k}) = \frac{e^{i\sigma\varphi_{\mathbf{k}}}}{k} = \frac{k_x + i\sigma k_y}{k^2}, \quad (\text{B.4})$$

into equation (B.3), and integrating over $\varphi_{\mathbf{k}}$ by using $k_x = k \cos \varphi_{\mathbf{k}}$ and $k_y = k \sin \varphi_{\mathbf{k}}$, we arrive at

$$\begin{aligned} \chi &= \frac{2\pi^2}{k^4} \mathcal{N} \int d\mathbf{X} d\mathbf{R} \sum_{\sigma} \left(\frac{\partial \mathcal{A}_{\sigma}^*}{\partial R_x} \frac{\partial \mathcal{A}_{\sigma}}{\partial R_x} - i\sigma \frac{\partial \mathcal{A}_{\sigma}^*}{\partial R_x} \frac{\partial \mathcal{A}_{\sigma}}{\partial R_y} + i\sigma \frac{\partial \mathcal{A}_{\sigma}^*}{\partial R_y} \frac{\partial \mathcal{A}_{\sigma}}{\partial R_x} + \frac{\partial \mathcal{A}_{\sigma}^*}{\partial R_y} \frac{\partial \mathcal{A}_{\sigma}}{\partial R_y} \right) \\ &= 2\pi^2 \sum_{\sigma} \mathcal{N} \int d\mathbf{X} d\mathbf{R} (\nabla_{\mathbf{R}} \mathcal{A}_{\sigma}^* \cdot \nabla_{\mathbf{R}} \mathcal{A}_{\sigma}) \cdot \frac{1}{k^4}, \end{aligned} \quad (\text{B.5})$$

where we have used

$$\int d\mathbf{R} \left(\frac{\partial \mathcal{A}_{\sigma}^*}{\partial R_x} \frac{\partial \mathcal{A}_{\sigma}}{\partial R_y} - \frac{\partial \mathcal{A}_{\sigma}^*}{\partial R_y} \frac{\partial \mathcal{A}_{\sigma}}{\partial R_x} \right) = 0. \quad (\text{B.6})$$

Appendix C. Calculation of $\nabla_{\mathbf{r}}^2 [N_m(qr) \Omega_m^{(\sigma)}(\varphi)]$

The Bessel function of the second kind $N_m(x)$ takes the following series power form at small x [42],

$$N_m(x) = -\frac{1}{\pi} \sum_{s=0}^{m-1} \frac{(m-s-1)!}{s!} \left(\frac{x}{2}\right)^{2s-m} + \frac{2}{\pi} \ln\left(\frac{x}{2}\right) J_m(x) - \frac{1}{\pi} \sum_{s=0}^{\infty} (-)^s \frac{\psi(s) + \psi(s+m+1)}{s!(s+m)!} \left(\frac{x}{2}\right)^{2s+m}, \quad (\text{C.1})$$

where $J_m(x)$ is the Bessel function of the first kind, and $\psi(\cdot)$ is the digamma function. Using the form of $\nabla_{\mathbf{r}}^2$ in the polar coordinate, we find

$$\nabla_{\mathbf{r}}^2 [N_m(qr) \Omega_m^{(\sigma)}(\varphi)] = \left[\left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{m^2}{r^2} \right) N_m(qr) \right] \Omega_m^{(\sigma)}(\varphi). \quad (\text{C.2})$$

For $m \geq 1$, we may separate the most singular term of $N_m(x)$ at small x , i.e. $-2^m(m-1)!/\pi x^m$, and find

$$\left(\frac{1}{x} \frac{\partial}{\partial x} x \frac{\partial}{\partial x} - \frac{m^2}{x^2} \right) \left[N_m(x) + \frac{2^m}{\pi} \cdot \frac{(m-1)!}{x^m} \right] = -N_m(x). \quad (\text{C.3})$$

Subsequently, we obtain

$$\nabla_{\mathbf{r}}^2 [N_m(qr) \Omega_m^{(\sigma)}(\varphi)] = \left\{ \left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{m^2}{r^2} \right) \left[-\frac{2^m}{\pi} \cdot \frac{(m-1)!}{(qr)^m} \right] - q^2 N_m(qr) \right\} \Omega_m^{(\sigma)}(\varphi). \quad (\text{C.4})$$

Using [27, 43]

$$\frac{d}{dr} \frac{1}{r^m} = -\frac{m}{r^{m+1}} + \frac{(-)^m}{m!} \delta^{(m)}(r), \quad (\text{C.5})$$

where $\delta^{(m)}(r)$ is the n th derivative of the Dirac delta function $\delta(r)$, and $\delta^{(m)}(r) = (-)^m m! \delta(r)/r^m$, we obtain

$$\left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{m^2}{r^2} \right) \left(\frac{1}{r^m} \right) = -\frac{2m}{r^{m+1}} \delta(r), \quad (\text{C.6})$$

and then

$$\nabla_{\mathbf{r}}^2 \left[N_m(qr) \Omega_m^{(\sigma)}(\varphi) \right] = \left[\frac{2^{m+1}}{\pi} \cdot \frac{m!}{q^m r^{m+1}} \delta(r) - q^2 N_m(qr) \right] \Omega_m^{(\sigma)}(\varphi), \quad (\text{C.7})$$

which finally yields equation (51).

Appendix D. Calculation details of equations (54) and (56)

In this part of the appendix, we present the calculation details of the derivations of equations (54) and (56). Let us look at equation (54) first. Using the identity (51), we easily find

$$\begin{aligned} & \frac{\pi^2 \hbar^2 q^2}{4M} \int_{r < \epsilon} \mathbf{d}\mathbf{r} N_1(qr) \Omega_1^{(\sigma')*}(\varphi) \nabla_{\mathbf{r}}^2 \left[N_1(qr) \Omega_1^{(\sigma)}(\varphi) \right] \\ &= \frac{\pi \hbar^2 q}{M} \int_{r < \epsilon} \mathbf{d}\mathbf{r} \left[N_1(qr) \Omega_1^{(\sigma')*}(\varphi) \right] \left[\frac{\delta(r)}{r^2} \Omega_1^{(\sigma)}(\varphi) \right] - \frac{\pi^2 \hbar^2 q^4}{4M} \\ & \int_{r < \epsilon} \mathbf{d}\mathbf{r} \left[N_1(qr) \Omega_1^{(\sigma')*}(\varphi) \right] \left[N_1(qr) \Omega_1^{(\sigma)}(\varphi) \right]. \end{aligned} \quad (\text{D.1})$$

Obviously, the two integrals of equation (D.1) are both divergent. Since we know

$$N_1(qr) = -\frac{2}{\pi qr} + \frac{qr}{\pi} \left(\ln \frac{qr}{2} + \gamma - \frac{1}{2} \right) + O(qr)^3 \quad (\text{D.2})$$

at $qr \sim 0$, the first integral of equation (D.1) becomes

$$\begin{aligned} & \frac{\pi \hbar^2 q}{M} \int_{r < \epsilon} \mathbf{d}\mathbf{r} \left[N_1(qr) \Omega_1^{(\sigma')*}(\varphi) \right] \left[\frac{\delta(r)}{r^2} \Omega_1^{(\sigma)}(\varphi) \right] \\ &= -\frac{2\hbar^2}{M} \int_{r < \epsilon} \mathbf{d}\mathbf{r} \frac{\Omega_1^{(\sigma')*}(\varphi)}{r} \left[\frac{\delta(r)}{r^2} \Omega_1^{(\sigma)}(\varphi) \right] + \frac{\hbar^2 q^2}{M} \int_{r < \epsilon} \mathbf{d}\mathbf{r} \left[r \ln \frac{qr}{2} \Omega_1^{(\sigma')*}(\varphi) \right] \\ & \left[\frac{\delta(r)}{r^2} \Omega_1^{(\sigma)}(\varphi) \right] + \delta_{\sigma\sigma'} \frac{\hbar^2 q^2}{2M} (2\gamma - 1). \end{aligned} \quad (\text{D.3})$$

Using the Fourier transform of $\frac{\delta(r)}{r^2} \Omega_1^{(\sigma)}(\varphi)$, i.e.

$$\mathcal{F} \left[\frac{\delta(r)}{r^2} \Omega_1^{(\sigma)}(\varphi) \right] = -i\sqrt{2\pi} e^{i\sigma\varphi} \frac{k}{2}, \quad (\text{D.4})$$

we obtain

$$\begin{aligned} & -\frac{2\hbar^2}{M} \int_{r < \epsilon} \mathbf{d}\mathbf{r} \frac{\Omega_1^{(\sigma')*}(\varphi)}{r} \left[\frac{\delta(r)}{r^2} \Omega_1^{(\sigma)}(\varphi) \right] = -\delta_{\sigma\sigma'} \frac{4\pi^2 \hbar^2}{M} \int \frac{kdk}{(2\pi)^2} [1 - J_0(k\epsilon)] \\ &= -\delta_{\sigma\sigma'} \frac{4\pi^2 \hbar^2}{M} \int \frac{kdk}{(2\pi)^2} \\ &= -\delta_{\sigma\sigma'} \frac{4\pi^2 \hbar^2}{M} \lim_{\Lambda \rightarrow \infty} \left[\int_0^\Lambda \frac{kdk}{(2\pi)^2} + \int_\Lambda^\infty \frac{kdk}{(2\pi)^2} \right] \\ &= \delta_{\sigma\sigma'} \lim_{\Lambda \rightarrow \infty} \left[-\frac{\hbar^2 \Lambda^2}{2M} - 8\pi^2 \int_\Lambda^\infty \frac{kdk}{(2\pi)^2} \frac{\hbar^2 k^2}{2M} \frac{1}{k^2} \right]. \end{aligned} \quad (\text{D.5})$$

Similarly,

$$\begin{aligned} \frac{\hbar^2 q^2}{M} \int_{r < \epsilon} \mathbf{dr} \left[r \ln \frac{qr}{2} \Omega_1^{(\sigma')*}(\varphi) \right] \left[\frac{\delta(r)}{r^2} \Omega_1^{(\sigma)}(\varphi) \right] &= \delta_{\sigma\sigma'} \frac{2\pi^2 \hbar^2 q^2}{M} \int \frac{kdk}{(2\pi)^2} \\ &\quad \frac{-2[1 - J_0(k\epsilon)] + k\epsilon J_1(k\epsilon) + [2J_1(k\epsilon) - k\epsilon J_0(k\epsilon)] k\epsilon \ln \frac{\epsilon q}{2}}{k^2}. \end{aligned} \quad (\text{D.6})$$

Since

$$\begin{aligned} &\frac{2\pi^2 \hbar^2 q^2}{M} \int \frac{kdk}{(2\pi)^2} \frac{-2[1 - J_0(k\epsilon)]}{k^2} \\ &= \frac{2\pi^2 \hbar^2 q^2}{M} \lim_{\Lambda \rightarrow \infty} \left[\int_0^\Lambda \frac{kdk}{(2\pi)^2} \frac{-2 + 2J_0(k\epsilon)}{k^2} + \int_\Lambda^\infty \frac{kdk}{(2\pi)^2} \frac{-2 + 2J_0(k\epsilon)}{k^2} \right] \\ &= \lim_{\Lambda \rightarrow \infty} \left[-\frac{\hbar^2 q^2}{M} \left(\gamma + \ln \frac{\epsilon \Lambda}{2} \right) - 8\pi^2 q^2 \int_\Lambda^\infty \frac{kdk}{(2\pi)^2} \frac{\hbar^2 k^2}{2M} \frac{1}{k^4} \right], \end{aligned} \quad (\text{D.7})$$

and

$$\frac{2\pi^2 \hbar^2 q^2}{M} \int \frac{kdk}{(2\pi)^2} \frac{k\epsilon J_1(k\epsilon) + [2J_1(k\epsilon) - k\epsilon J_0(k\epsilon)] k\epsilon \ln \frac{\epsilon q}{2}}{k^2} = \frac{\hbar^2 q^2}{M} \left(\frac{1}{2} + \ln \frac{\epsilon q}{2} \right), \quad (\text{D.8})$$

we find

$$\begin{aligned} \frac{\hbar^2 q^2}{M} \int_{r < \epsilon} \mathbf{dr} \left[r \ln \frac{qr}{2} \Omega_1^{(\sigma')*}(\varphi) \right] \left[\frac{\delta(r)}{r^2} \Omega_1^{(\sigma)}(\varphi) \right] \\ = \delta_{\sigma\sigma'} \lim_{\Lambda \rightarrow \infty} \left[-\frac{\hbar^2 q^2}{2M} (2\gamma - 1) - \frac{\hbar^2 q^2}{M} \ln \frac{\Lambda}{q} - 8\pi^2 q^2 \int_\Lambda^\infty \frac{kdk}{(2\pi)^2} \frac{\hbar^2 k^2}{2M} \frac{1}{k^4} \right], \end{aligned} \quad (\text{D.9})$$

and then

$$\begin{aligned} \frac{\pi \hbar^2 q}{M} \int_{r < \epsilon} \mathbf{dr} \left[N_1(qr) \Omega_1^{(\sigma')*}(\varphi) \right] \left[\frac{\delta(r)}{r^2} \Omega_1^{(\sigma)}(\varphi) \right] \\ = \delta_{\sigma\sigma'} \lim_{\Lambda \rightarrow \infty} \left[-\frac{\hbar^2 \Lambda^2}{2M} - \frac{\hbar^2 q^2}{M} \ln \frac{\Lambda}{q} - \int_\Lambda^\infty \frac{kdk}{(2\pi)^2} \frac{\hbar^2 k^2}{2M} \left(\frac{8\pi^2}{k^2} + \frac{8\pi^2 q^2}{k^4} \right) \right]. \end{aligned} \quad (\text{D.10})$$

As to the second term of equation (D.1), we have

$$\begin{aligned} &-\frac{\pi^2 \hbar^2 q^4}{4M} \int_{r < \epsilon} \mathbf{dr} \left[N_1(qr) \Omega_1^{(\sigma')*}(\varphi) \right] \left[N_1(qr) \Omega_1^{(\sigma)}(\varphi) \right] \\ &= -\frac{\hbar^2 q^2}{M} \left[\int_{r < \epsilon} \mathbf{dr} \frac{\Omega_1^{(\sigma')*}(\varphi)}{r} \frac{\Omega_1^{(\sigma)}(\varphi)}{r} + O(\epsilon q)^2 \right]. \end{aligned} \quad (\text{D.11})$$

Using the Fourier transform of $\Omega_1^{(\sigma)}(\varphi)/r$, i.e.

$$\mathcal{F} \left[\frac{\Omega_1^{(\sigma)}(\varphi)}{r} \right] = -i\sqrt{2\pi} \frac{e^{i\sigma\varphi\mathbf{k}}}{k}, \quad (\text{D.12})$$

we obtain

$$\begin{aligned}
& -\frac{\pi^2 \hbar^2 q^4}{4M} \int_{r < \epsilon} \mathbf{dr} \left[N_1(qr) \Omega_1^{(\sigma')*}(\varphi) \right] \left[N_1(qr) \Omega_1^{(\sigma)}(\varphi) \right] \\
& = -\delta_{\sigma\sigma'} \frac{4\pi^2 \hbar^2 q^2}{M} \int \frac{kdk}{(2\pi)^2} \frac{1 - J_0(k\epsilon)}{k^2} \\
& = -\delta_{\sigma\sigma'} \frac{4\pi^2 \hbar^2 q^2}{M} \lim_{\Lambda \rightarrow \infty} \left[\int_0^\Lambda \frac{kdk}{(2\pi)^2} \frac{1 - J_0(k\epsilon)}{k^2} + \int_\Lambda^\infty \frac{kdk}{(2\pi)^2} \frac{1 - J_0(k\epsilon)}{k^2} \right] \\
& = \delta_{\sigma\sigma'} \lim_{\Lambda \rightarrow \infty} \left[-\frac{\hbar^2 q^2}{M} \left(\gamma + \ln \frac{\epsilon\Lambda}{2} \right) - \int_\Lambda^\infty \frac{kdk}{(2\pi)^2} \frac{\hbar^2 k^2}{2M} \frac{8\pi^2 q^2}{k^4} \right] \quad (\text{D.13})
\end{aligned}$$

in the limit $\epsilon q \rightarrow 0$. Combining equations (D.10) and (D.13), we finally obtain equation (54) in the maintext, i.e.

$$\begin{aligned}
& \frac{\pi^2 \hbar^2 q^2}{4M} \int_{r < \epsilon} \mathbf{dr} N_1(qr) \Omega_1^{(\sigma')*}(\varphi) \nabla_{\mathbf{r}}^2 \left[N_1(qr) \Omega_1^{(\sigma)}(\varphi) \right] \\
& = \delta_{\sigma\sigma'} \lim_{\Lambda \rightarrow \infty} \left[-\frac{\hbar^2 \Lambda^2}{2M} - \frac{\hbar^2 q^2}{M} \ln \frac{\Lambda}{q} \right. \\
& \quad \left. - \frac{\hbar^2 q^2}{M} \left(\gamma + \ln \frac{\epsilon\Lambda}{2} \right) - \int_\Lambda^\infty \frac{kdk}{(2\pi)^2} \frac{\hbar^2 k^2}{2M} \left(\frac{8\pi^2}{k^2} + \frac{16\pi^2 q^2}{k^4} \right) \right]. \quad (\text{D.14})
\end{aligned}$$

In the follows, let us look at equation (56). It is easily found

$$\mathbf{I}_2^{(\sigma\sigma')} = \int_{r < \epsilon} \mathbf{dr} \frac{\Omega_1^{(\sigma')*}(\varphi)}{r} \frac{\Omega_1^{(\sigma)}(\varphi)}{r} + \mathcal{O}(\epsilon q)^2. \quad (\text{D.15})$$

Using the Fourier transform of $\Omega_1^{(\sigma)}(\varphi)/r$, i.e. equation (D.12), we have

$$\mathbf{I}_2^{(\sigma\sigma')} = \delta_{\sigma\sigma'} 4\pi^2 \int \frac{kdk}{(2\pi)^2} \frac{1 - J_0(k\epsilon)}{k^2} \quad (\text{D.16})$$

in the limit $\epsilon q \rightarrow 0$, and then $\mathbf{I}_2^{(\sigma\sigma')}$ can alternatively be written as

$$\begin{aligned}
\mathbf{I}_2^{(\sigma\sigma')} & = \delta_{\sigma\sigma'} \lim_{\Lambda \rightarrow \infty} \left[4\pi^2 \int_0^\Lambda \frac{kdk}{(2\pi)^2} \frac{1 - J_0(k\epsilon)}{k^2} + 4\pi^2 \int_\Lambda^\infty \frac{kdk}{(2\pi)^2} \frac{1 - J_0(k\epsilon)}{k^2} \right] \\
& = \delta_{\sigma\sigma'} \lim_{\Lambda \rightarrow \infty} \left[\gamma + \ln \frac{\epsilon\Lambda}{2} + 4\pi^2 \int_\Lambda^\infty \frac{kdk}{(2\pi)^2} \frac{1}{k^2} \right], \quad (\text{D.17})
\end{aligned}$$

where we have used

$$\lim_{\Lambda \rightarrow \infty} \int_\Lambda^\infty \frac{kdk}{(2\pi)^2} \frac{J_0(k\epsilon)}{k^2} = 0. \quad (\text{D.18})$$

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