

## Contact Theory for Spin-Orbit-Coupled Fermi Gases

Shi-Guo Peng,<sup>1</sup> Cai-Xia Zhang,<sup>1,4</sup> Shina Tan,<sup>2,3,\*</sup> and Kaijun Jiang<sup>1,3,†</sup>

<sup>1</sup>State Key Laboratory of Magnetic Resonance and Atomic and Molecular Physics, Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, Wuhan 430071, China

<sup>2</sup>School of Physics, Georgia Institute of Technology, Atlanta, Georgia 30332, USA

<sup>3</sup>Center for Cold Atom Physics, Chinese Academy of Sciences, Wuhan 430071, China

<sup>4</sup>School of Physics, University of Chinese Academy of Sciences, Beijing 100049, China



(Received 29 October 2017; published 9 February 2018)

We develop the contact theory for spin-orbit-coupled Fermi gases. By using a perturbation method, we derive analytically the universal two-body behavior at short distance, which does not depend on the short-range details of interatomic potentials. We find that two new scattering parameters need to be introduced because of spin-orbit coupling, besides the traditional *s*- and *p*-wave scattering length (volume) and effective ranges. This is a general and unique feature for spin-orbit-coupled systems. Consequently, two new adiabatic energy relations with respect to the new scattering parameters are obtained, in which a new contact is involved because of spin-orbit coupling. In addition, we derive the asymptotic behavior of the large-momentum distribution, and find that the subleading tail is corrected by the new contact. This work paves the way for exploring the profound properties of spin-orbit-coupled many-body systems, according to two-body solutions.

DOI: [10.1103/PhysRevLett.120.060408](https://doi.org/10.1103/PhysRevLett.120.060408)

*Introduction.*—Universality, referring to observations independent of short-range details, is one of the most fascinating and intriguing phenomena in modern physics. In ultracold atoms, a set of universal relations, following from the short-range behavior of the two-body physics are discovered [1]. These relations are connected simply by a universal contact parameter, which overarches between microscopic and macroscopic properties of a strongly interacting many-body system. Nowadays, the contact theory becomes significantly important in ultracold atomic physics, and has systematically been verified and investigated both experimentally and theoretically [2–7]. Nevertheless, the contact theory for spin-orbit-coupled systems is still unexplored until now, even though the spin-orbit (SO) coupling was realized in cold atoms several years ago [8–10], and the resulting unique phenomena have attracted a great deal of interest, such as topological insulators and superconductors [11–14].

In this Letter, for the first time, we generalize the contact theory to strongly interacting spin-orbit-coupled Fermi gases, and the single-particle Hamiltonian takes the form,

$$\hat{\mathcal{H}}_1 = \frac{\hbar^2 \hat{\mathbf{k}}_1^2}{2M} + \frac{\hbar^2 \lambda}{M} \hat{\mathbf{k}}_1 \cdot \hat{\boldsymbol{\sigma}} + \frac{\hbar^2 \lambda^2}{2M}, \quad (1)$$

where  $\hat{\mathbf{k}}_1 = -i\nabla$  and  $\hat{\boldsymbol{\sigma}}$  are respectively the single-particle momentum and spin operators,  $\lambda > 0$  is the strength of SO coupling,  $M$  is the atomic mass, and  $\hbar$  is the Planck's constant divided by  $2\pi$ . Here, the SO coupling is assumed to be isotropic for simplicity, and the possible scheme for

the realization of the three-dimensional (3D) isotropic SO coupling is proposed in Ref. [15]. Because of SO coupling, the orbital angular momentum of the relative motion of two fermions is no longer conserved, and then all the partial-wave scatterings are coupled [16]. Fortunately, the total momentum  $\mathbf{K}$  of two fermions is still conserved as well as the total angular momentum  $\mathbf{J}$ . Therefore, we may reasonably focus on the two-body problem in the subspace of  $\mathbf{K} = 0$  and  $\mathbf{J} = 0$  for simplicity, and then only *s*- and *p*-wave scatterings are coupled [16,17]. Consequently, the two spin-half fermions in the subspace of  $\mathbf{K} = 0$  and  $\mathbf{J} = 0$  is described by the following two-body Hamiltonian

$$\hat{\mathcal{H}}_2 = \frac{\hbar^2 \hat{\mathbf{k}}^2}{M} + \frac{\hbar^2 \lambda}{M} \hat{\mathbf{k}} \cdot (\hat{\boldsymbol{\sigma}}_2 - \hat{\boldsymbol{\sigma}}_1) + \frac{\hbar^2 \lambda^2}{M} + V(r), \quad (2)$$

where  $\hat{\mathbf{k}}$  is the momentum operator for the relative motion  $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ ,  $\hat{\boldsymbol{\sigma}}_i$  is the spin operator of the *i*th atom, and  $V(r)$  is the interatomic potential with a finite range  $\epsilon$ . Our theory may also be generalized to the case of  $\mathbf{K} \neq 0$  and  $\mathbf{J} \neq 0$ , and then more partial waves should be involved.

One of the most daunting challenges for establishing the contact theory is how to obtain the *universal* two-body behavior at short distance for a SO-coupled Fermi gas. Although the SO-coupled two-body problem was considered recently by using a spherical square-well potential [16–18], the general form of such universal behavior for any interatomic potential still remains elusive until now. In this work, we develop a perturbation method to construct the short-range asymptotic form of the two-body wave

function for a SO-coupled system. We find that two new scattering parameters  $u$ ,  $v$  need to be introduced in the short-range behavior of two-body wave functions, besides the traditional scattering length (volume) and effective ranges. The obtained universal behavior does not depend on the short-range details of the interatomic potentials, and thus is feasible for any interatomic potential with short range. Two new adiabatic energy relations are accordingly found with respect to the new scattering parameters, i.e.,

$$\frac{\partial E}{\partial u} = \frac{\hbar^2 \lambda}{32\pi^2 M} \left( C_a^{(0)} - \lambda \frac{\mathcal{P}_\lambda}{2} \right), \quad (3)$$

$$\frac{\partial E}{\partial v} = \lambda \frac{3\hbar^2 C_a^{(1)}}{32\pi^2 M}, \quad (4)$$

in which we hold all the other two-body parameters unchanged in the partial derivatives. Here,  $C_a^{(0)}$  is the well-known  $s$ -wave contact and  $C_a^{(1)}$  is the  $p$ -wave contact corresponding to the  $p$ -wave scattering volume [3,6]. In addition,  $\mathcal{P}_\lambda$  is the new contact introduced by SO coupling. Further, we derive the asymptotic behavior of the large-momentum distribution from the universal two-body behavior at short distance,

$$n(q) = \frac{C_a^{(1)}}{q^2} + (C_a^{(0)} + C_b^{(1)} + \lambda \mathcal{P}_\lambda) \frac{1}{q^4} + \mathcal{O}(q^{-6}), \quad (5)$$

in which  $C_b^{(1)}$  is the  $p$ -wave contact corresponding to the  $p$ -wave effective range. We find that the subleading tail ( $q^{-4}$ ) of the large-momentum distribution is amended by the new contact  $\mathcal{P}_\lambda$  because of SO coupling.

*Universal short-range behavior of two-body wave functions.*—Let us consider the two-body problem of a SO-coupled system in the subspace of  $\mathbf{K} = 0$  and  $\mathbf{J} = 0$ , and the corresponding Hamiltonian takes the form of Eq. (2). The subspace is spanned by two orthogonal basis, i.e.,  $\Omega_0(\hat{\mathbf{r}}) = Y_{00}(\hat{\mathbf{r}})|S\rangle$  and  $\Omega_1(\hat{\mathbf{r}}) = -i[Y_{1-1}(\hat{\mathbf{r}})|\uparrow\uparrow\rangle + Y_{11}(\hat{\mathbf{r}})|\downarrow\downarrow\rangle - Y_{10}(\hat{\mathbf{r}})|T\rangle]/\sqrt{3}$ , where  $Y_{lm}(\hat{\mathbf{r}})$  is the spherical harmonics,  $\hat{\mathbf{r}}$  denotes the angular part of the coordinate  $\mathbf{r}$ , and  $|S\rangle = [(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}]$  and  $\{|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, |T\rangle = (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}\}$  are the spin-singlet and spin-triplet states with total spin 0 and 1, respectively. The two-body solution can formally be written in the basis of  $\{\Omega_0(\hat{\mathbf{r}}), \Omega_1(\hat{\mathbf{r}})\}$  as  $\Psi(\mathbf{r}) = \psi_0(r)\Omega_0(\hat{\mathbf{r}}) + \psi_1(r)\Omega_1(\hat{\mathbf{r}})$  [16,17].

Since the SO effect exists even inside the interatomic potential, it should modify the short-range behavior of the two-body wave function dramatically [19]. However, in current experiments of ultracold atoms [20], the SO-coupling strength  $\lambda$  is of the order  $\mu\text{m}^{-1}$ , pretty small compared to the inverse of the range of interatomic potential  $\epsilon^{-1}$  (of the order  $\text{nm}^{-1}$ ). Moreover, the momentum

$k = \sqrt{ME/\hbar^2}$  is also much smaller than  $\epsilon^{-1}$  in the low-energy scattering limit. Therefore, when two fermions get as close as the range  $\epsilon$ , we may deal with the SO coupling perturbatively as well as the energy, and assume that the form of the two-body solution has the following structure:

$$\Psi(\mathbf{r}) \approx \phi(\mathbf{r}) + k^2 f(\mathbf{r}) - \lambda g(\mathbf{r}), \quad (6)$$

as  $r \sim \epsilon$ . Here, we keep up to the first-order terms of  $k^2$  and  $\lambda$ . The advantage of this ansatz is that the functions  $\phi(\mathbf{r})$ ,  $f(\mathbf{r})$ , and  $g(\mathbf{r})$  are all independent on the energy and SO-coupling strength. Therefore, they are determined only by the short-range details of the interaction, and characterize the intrinsic properties of the interatomic potential. We expect that the traditional scattering length or volume in the absence of SO coupling are included in the zero-order term  $\phi(\mathbf{r})$ , while the effective ranges are involved in  $f(\mathbf{r})$ , the coefficient of the first-order term of  $k^2$ . Interestingly, new scattering parameters should appear in the first-order term of  $\lambda$  [in  $g(\mathbf{r})$ ], which are introduced by SO coupling. Conveniently, more scattering parameters may be introduced if higher-order terms of  $k^2$  and  $\lambda$  are perturbatively considered. Inserting the ansatz (6) into the Schrödinger equation, and comparing the corresponding coefficients of  $k^2$  and  $\lambda$ , we obtain

$$\left( -\nabla^2 + \frac{M}{\hbar^2} V(r) \right) \phi(\mathbf{r}) = 0, \quad (7)$$

$$\left( -\nabla^2 + \frac{M}{\hbar^2} V(r) \right) f(\mathbf{r}) = \phi(\mathbf{r}), \quad (8)$$

$$\left( -\nabla^2 + \frac{M}{\hbar^2} V(r) \right) g(\mathbf{r}) = Q(\mathbf{r})\phi(\mathbf{r}), \quad (9)$$

where  $Q(\mathbf{r}) = -i\nabla \cdot (\hat{\boldsymbol{\sigma}}_2 - \hat{\boldsymbol{\sigma}}_1)$ . These equations can analytically be solved for  $r > \epsilon$ , and simply yield

$$\begin{aligned} \Psi(\mathbf{r}) = & \alpha_0 \left[ \frac{1}{r} + \left( -\frac{1}{a_0} + \frac{b_0}{2} k^2 + u\lambda \right) - \frac{k^2}{2} r \right] \Omega_0(\hat{\mathbf{r}}) \\ & + \alpha_1 \left[ \frac{1}{r^2} + \left( \frac{k^2}{2} + \frac{\alpha_0}{\alpha_1} \lambda \right) \right. \\ & \left. + \left( -\frac{1}{3a_1} + \frac{b_1}{6} k^2 + v\lambda \right) r \right] \Omega_1(\hat{\mathbf{r}}) + \mathcal{O}(r^2), \quad (10) \end{aligned}$$

where  $\alpha_0$  and  $\alpha_1$  are two complex superposition coefficients. Apparently,  $a_0$ ,  $b_0$  are the  $s$ -wave scattering length and effective range, and  $a_1$ ,  $b_1$  are the  $p$ -wave scattering volume and effective range without SO coupling, respectively. For simplicity, we may only consider the case with  $b_0 \approx 0$  for broad  $s$ -wave resonances throughout the Letter. We can see that the  $s$ -wave component is hybridized in the  $p$ -wave channel by SO coupling as manifested as the term  $\alpha_0 \lambda / \alpha_1$ . Interestingly, two new scattering parameters  $u$  and

$v$  as we anticipate are involved. They are the corrections from SO coupling to the short-range behavior of the two-body wave function in  $s$ - and  $p$ -wave channels, respectively. If  $\lambda = 0$ , the  $s$ - and  $p$ -wave scatterings decouple, and the asymptotic form of  $\Psi(\mathbf{r})$  at small  $\mathbf{r}$ , i.e., Eq. (10), simply reduces to the ordinary  $s$ - and  $p$ -wave short-range boundary conditions, respectively. The derivation above does not depend on the short-range details of the interaction, and thus is universal and applicable for all kinds of neutral fermionic atoms.

In general, the  $s$ - and  $p$ -wave scatterings in different spin channels should both be taken into account because of SO coupling. We may roughly estimate which partial wave is more important as follows. Without SO coupling, and away from any resonances—in the weak interacting limit, the two-body wave function should well behave as  $\mathbf{r} \rightarrow 0$  as  $\Psi(\mathbf{r}) \sim (\alpha_0/a_0)\Omega_0(\hat{\mathbf{r}}) + (\alpha_1 r/3a_1)\Omega_1(\hat{\mathbf{r}})$ . If we assume that the atoms are initially prepared equally in the spin channels  $\Omega_0(\hat{\mathbf{r}})$  and  $\Omega_1(\hat{\mathbf{r}})$ , we have  $\alpha_0/\alpha_1 \sim a_0 r/3a_1$ . When interatomic interactions are turned on, the two-body wave function becomes divergent as  $\mathbf{r} \rightarrow 0$  ( $> \epsilon$ ),  $\alpha_0 r^{-1}$  and  $\alpha_1 r^{-2}$  for  $s$ - and  $p$ -wave scatterings, respectively. This divergent behavior is unchanged even in the presence of SO coupling. Then the ratio between the strengths of  $s$ - and  $p$ -wave scatterings at small  $\mathbf{r}$  becomes  $(\alpha_0 r^{-1})/(\alpha_1 r^{-2}) \approx a_0 r^2/3a_1$ . Near  $s$ -wave resonances, we have  $a_0 \sim k_f^{-1}$ ,  $a_1 \sim \epsilon^3$ ,  $r \sim \epsilon$ , where  $k_f$  is the Fermi wave number, and then this ratio is approximately of the order  $(k_f \epsilon)^{-1} \gg 1$ . Therefore, the  $s$ -wave interaction dominates the two-body scattering. By noticing  $\Omega_0(\hat{\mathbf{r}}) = |S\rangle/\sqrt{4\pi}$ , and  $\Omega_1(\hat{\mathbf{r}}) = -i(\hat{\boldsymbol{\sigma}}_2 - \hat{\boldsymbol{\sigma}}_1) \cdot (\mathbf{r}/r)|S\rangle/\sqrt{16\pi}$ , and if the  $p$ -wave interaction could be ignored near broad  $s$ -wave resonances, Eq. (10) becomes (up to a prefactor  $\alpha_0/\sqrt{4\pi}$ ),

$$\Psi(\mathbf{r}) = \left(\frac{1}{r} - \frac{1}{a_0} + u\lambda\right)|S\rangle - i\frac{\lambda}{2}(\hat{\boldsymbol{\sigma}}_2 - \hat{\boldsymbol{\sigma}}_1) \cdot \frac{\mathbf{r}}{r}|S\rangle + \mathcal{O}(r), \quad (11)$$

which exactly recovers the result of Ref. [19] [see Eq. (31) of Ref. [19]] with  $a_r^{-1} = a_0^{-1} - u\lambda$ .

Near  $p$ -wave resonances, for example, the  $p$ -wave Feshbach resonance at  $B_0 = 185.09$  G in  ${}^6\text{Li}$  [21], we have  $a_0 \sim \epsilon$ ,  $a_1 \sim k_f^{-3}$ ,  $r \sim \epsilon$ , then the ratio between the strengths of  $s$ - and  $p$ -wave scatterings is roughly of the order  $(k_f \epsilon)^3 \ll 1$ . In this case, the  $p$ -wave scattering becomes significantly important.

*Large-momentum distribution.*—For a many-body system with  $N$  spin-half fermions, if only two-body correlations are taken into account, the many-body wave function  $\Psi_N$  can approximately be written as the form of Eq. (10), when fermions  $(i, j)$  get close while all the others are far away. In this case,  $\mathbf{r} = \mathbf{r}_i - \mathbf{r}_j$ , and the arbitrary complex numbers  $\alpha_0$  and  $\alpha_1$  become the functions of the variable  $\mathbf{X}$ ,

which involves both the center-of-mass (c.m.) coordinate of the pair being considered and the coordinates of all the other fermions. Further,  $\alpha_0$  and  $\alpha_1$  should be constrained by the normalization of the many-body wave function. Using the asymptotic form of the many-body wave function  $\Psi_N$  at small  $\mathbf{r}$ , we can easily obtain the behavior of the tail of the single-particle momentum distribution at large  $\mathbf{q}$  (but smaller than  $\epsilon^{-1}$ ), which is defined as  $n(\mathbf{q}) \equiv \sum_{i=1}^N \int \prod_{j \neq i} d\mathbf{r}_j \int d\mathbf{r}_i \Psi_N e^{-i\mathbf{q} \cdot \mathbf{r}_i} |^2$ .

After straightforward algebra, we easily obtain the momentum distribution  $n(q)$  taking the form of Eq. (5) at large  $q$  ( $< \epsilon^{-1}$ ). Here, we are only interested in the dependence of the momentum distribution on the amplitude of  $\mathbf{q}$ , and have already integrated over the angular part of  $\mathbf{q}$ . We find that

$$C_a^{(\nu)} = 32\pi^2 \mathcal{N} \int d\mathbf{X} |\alpha_\nu(\mathbf{X})|^2, \quad (\nu = 0, 1), \quad (12)$$

$$C_b^{(1)} = \frac{64\pi^2 M \mathcal{N}}{\hbar^2} \int d\mathbf{X} \alpha_1^*(\mathbf{X}) [E - \hat{T}(\mathbf{X})] \alpha_1(\mathbf{X}) \quad (13)$$

are the conventional  $s$ - and  $p$ -wave contacts [6], where  $\hat{T}(\mathbf{X})$  denotes the operators of the c.m. motion of the pair  $(i, j)$  and all the other fermions, and  $\mathcal{N} = N(N-1)/2$  is the number of all the possible ways to pair atoms. Besides, a new contact  $\mathcal{P}_\lambda$  resulting from SO coupling appears, which is defined as

$$\mathcal{P}_\lambda \equiv 64\pi^2 \mathcal{N} \int d\mathbf{X} \alpha_0^*(\mathbf{X}) \alpha_1(\mathbf{X}) + \text{c.c.} \quad (14)$$

Obviously, this new contact describes the interplay of the  $s$ - and  $p$ -wave scatterings because of SO coupling.

Since the momentum distribution at large  $\mathbf{q}$  is only characterized by the short-range behavior of the two-body physics, we may roughly estimate the order of all the quantities in the large- $\mathbf{q}$  behavior of the momentum distribution simply according to the two-body picture as before. Near  $s$ -wave resonances, if initially without SO coupling and away from any resonances, the atoms are prepared equally in the spin states  $\Omega_0(\hat{\mathbf{r}})$  and  $\Omega_1(\hat{\mathbf{r}})$ , we have  $\alpha_0/\alpha_1 \sim a_0 r/3a_1$ , and then  $C_a^{(1)} q^{-2}/C_a^{(0)} q^{-4} \approx 9a_1^2 q^2/a_0^2 r^2$ , which is roughly of the order  $(k_f \epsilon)^4 \ll 1$ . Besides, we may also find  $C_b^{(1)}/C_a^{(0)} \sim (k_f \epsilon)^4 \ll 1$ . This means that the  $p$ -wave contribution to the tail of momentum distribution at large  $\mathbf{q}$  may reasonably be ignored, which is consistent with the discussion before. However, the SO-coupling correction is notable compared to the  $p$ -wave contact in the subleading tail of the momentum distribution, i.e.,  $\lambda \mathcal{P}_\lambda/C_b^{(1)} \sim (k_f \epsilon)^{-2} \gg 1$ .

Near  $p$ -wave resonances, the leading tail  $q^{-2}$  of the large-momentum distribution becomes important, because  $C_a^{(1)} q^{-2}/C_a^{(0)} q^{-4} \sim (k_f \epsilon)^{-4} \gg 1$ . In the subleading tail of

$q^{-4}$ , we find  $C_b^{(1)}/C_a^{(0)} \sim (k_f \epsilon)^{-4} \gg 1$ ; thus, the  $s$ -wave contribution may be ignored. Consequently, the momentum distribution at large  $\mathbf{q}$  behaves as  $C_a^{(1)} q^{-2} + (C_b^{(1)} + \lambda \mathcal{P}_\lambda) q^{-4}$  with a considerable correction of  $\lambda \mathcal{P}_\lambda$  in the subleading tail due to SO coupling, compared to the  $s$ -wave contribution, i.e.,  $\lambda \mathcal{P}_\lambda / C_a^{(0)} \sim (k_f \epsilon)^{-2} \gg 1$ .

*Adiabatic energy relations.*—The thermodynamics of many-body systems, which is seemingly uncorrelated to the momentum distribution, is also characterized by the contacts defined above. A set of adiabatic energy relations describe how the energy of a many-body system changes as the two-body interaction is adiabatically adjusted. Let us consider two many-body wave functions  $\Psi_N$  and  $\Psi'_N$  corresponding to different interatomic interaction strengths. From the Schrödinger equations satisfied by  $\Psi_N$  and  $\Psi'_N$ , we easily obtain

$$(E - E') \iiint_{\mathcal{D}_\epsilon} d\mathbf{r}_1 d\mathbf{r}_2 \cdots d\mathbf{r}_N \Psi_N^* \Psi'_N = -\frac{\hbar^2 \mathcal{N}}{M} \oint_{\mathbb{F}=\epsilon} \mathbf{I} \cdot \hat{\mathbf{n}} d\Sigma + \frac{\hbar^2 \lambda \mathcal{N}}{2\pi M} \oint_{\mathbb{F}=\epsilon} \mathbf{F} \cdot \hat{\mathbf{n}} d\Sigma, \quad (15)$$

where  $\mathbf{I} \equiv \Psi'^* \nabla \Psi - (\nabla \Psi'^*) \Psi$ ,  $\mathbf{F} \equiv (\psi_1'^* \psi_0 - \psi_0^* \psi_1) \hat{\mathbf{e}}_r$  with the unit radial vector  $\hat{\mathbf{e}}_r$  of  $\mathbf{r}$ , and  $\Psi_N(\mathbf{X}, \mathbf{r}) = \psi_0(\mathbf{X}, r) \Omega_0(\hat{\mathbf{r}}) + \psi_1(\mathbf{X}, r) \Omega_1(\hat{\mathbf{r}})$ . Here, the domain  $\mathcal{D}_\epsilon$  is the set of all configurations  $(\mathbf{r}_i, \mathbf{r}_j)$  with  $r = |\mathbf{r}_i - \mathbf{r}_j| > \epsilon$ ,  $\Sigma$  is the surface in which the distance between the two atoms in the pair  $(i, j)$  is  $\epsilon$ , and  $\hat{\mathbf{n}}$  is the direction normal to  $\Sigma$  but is opposite to the radial direction. Using the asymptotic form of the many-body wave function  $\Psi_N$  at small  $\mathbf{r}$ , we find

$$\delta E = -\frac{\hbar^2}{32\pi^2 M} \left[ \left( C_a^{(0)} - \lambda \frac{\mathcal{P}_\lambda}{2} \right) \delta a_0^{-1} + C_a^{(1)} \delta a_1^{-1} - \frac{C_b^{(1)}}{4} \delta b_1 - \lambda \left( C_a^{(0)} - \lambda \frac{\mathcal{P}_\lambda}{2} \right) \delta u - 3\lambda C_a^{(1)} \delta v \right], \quad (16)$$

which characterizes how the energy of the system varies as the scattering parameters adiabatically change. In the absence of SO coupling, Eq. (16) simply reduces to the ordinary form of the adiabatic energy relations for  $s$ - and  $p$ -wave interactions [6,22], with respect to the scattering length (or volume) as well as effective range. Because of SO coupling, two new scattering parameters come into the problem, and then additional new adiabatic energy relations appear, i.e., Eqs. (3)–(4). These adiabatic energy relations demonstrate how the macroscopic thermodynamics of SO-coupled many-body systems varies with microscopic two-body scattering parameters.

*Contacts in a two-body problem.*—On behalf of the future experiments and calculations, we may explicitly evaluate the contacts defined above for a two-body bound

state, the wave function of which may be written as a column vector in the basis of  $\{\Omega_0(\hat{\mathbf{r}}), \Omega_1(\hat{\mathbf{r}})\}$  as [17]

$$\Psi_b(\mathbf{r}) = B\kappa_- \begin{bmatrix} h_0^{(1)}(\kappa_- r) \\ -h_1^{(1)}(\kappa_- r) \end{bmatrix} + D\kappa_+ \begin{bmatrix} h_0^{(1)}(\kappa_+ r) \\ h_1^{(1)}(\kappa_+ r) \end{bmatrix}, \quad (17)$$

where  $\kappa_\pm = i\kappa \pm \lambda$ , and  $\kappa = \sqrt{-ME/\hbar^2}$ . The binding energy  $E$  can be determined by expanding  $\Psi_b(\mathbf{r})$  at small  $\mathbf{r}$  and comparing with the short-range boundary condition (10), then the two-body contacts are easily obtained according to the adiabatic relations. Near  $s$ -wave resonances, we find

$$E = -\frac{\hbar^2}{Ma_0^2} + \frac{2\hbar^2 u}{Ma_0} \lambda + O(\lambda^2), \quad (18)$$

which simply reduces to the result  $E = -\hbar^2/Ma_0^2$  in the absence of SO coupling. Then we immediately obtain  $C_a^{(0)} = 64\pi^2/a_0$  and  $\mathcal{P}_\lambda = 128\pi^2 u$  by using adiabatic relations. Near  $p$ -wave resonances, we find

$$E = \frac{2\hbar^2}{Ma_1 b_1} - \frac{6\hbar^2 v}{Mb_1} \lambda + O(\lambda^2), \quad (19)$$

which is consistent with that without SO coupling [3], and then it yields  $C_a^{(1)} = -64\pi^2/b_1$  and  $C_b^{(1)} = -256\pi^2(a_1^{-1} - 3v\lambda)/b_1^2$ .

*Grand canonical potential and pressure relation.*—The adiabatic relations as well as the large-momentum distribution we obtained above is valid for any pure energy eigenstate. Therefore, they should still hold for any incoherent mixed state statistically at finite temperature. Then the energy, particle number density, and contacts become their statistical average values. It should be interesting to discuss how the results presented above affect the finite-temperature thermodynamics. To this end, let us look at the grand canonical potential, which is defined as  $\mathcal{J} \equiv -PV$  [23], where  $P$  is the pressure and  $V$  is the volume of the system. According to straightforward dimensional analysis [24,25], we can obtain

$$\mathcal{J} = -\frac{2}{3} E - \frac{\hbar^2}{96\pi^2 Ma_0} \left( C_a^{(0)} - \lambda \frac{\mathcal{P}_\lambda}{2} \right) - \frac{\hbar^2 C_a^{(1)}}{32\pi^2 Ma_1} + \frac{\hbar^2 b_1 C_b^{(1)}}{384\pi^2 M} + \lambda \frac{\hbar^2 v C_a^{(1)}}{16\pi^2 M}, \quad (20)$$

which alternatively yields the pressure relation by dividing both sides of Eq. (20) by  $-V$ .

*Conclusions.*—We systematically study the contact theory for spin-orbit-coupled Fermi gases. The universal two-body behavior at short distance is analytically derived, by introducing a perturbation method, which does not depend on the short-range details of interatomic potentials.

For simplicity, we focus on the  $s$ - and  $p$ -wave scatterings in the subspace of vanishing center-of-mass momentum and total angular momentum. Interestingly, two new microscopic scattering parameters appear in the short-range behavior of two-body wave functions because of spin-orbit coupling. We claim that this is a general and unique feature for spin-orbit-coupled systems, and thus the obtained universal short-range behavior of two-body wave functions is feasible for all kinds of neutral fermionic atoms. Consequently, a new contact is introduced originated from spin-orbit coupling, which, combining with conventional  $s$ - and  $p$ -wave contacts, characterizes the universal properties of spin-orbit-coupled many-body systems. In general, more partial-wave scatterings should be taken into account for nonzero center-of-mass momentum and nonzero total angular momentum. Then more contacts should appear. Our method could conveniently be generalized to other kinds of spin-orbit couplings as well as to low dimensions. Besides, our method could also be applied to bosons. In the presence of spin-orbit coupling, we expect that additional contacts would be introduced for bosonic systems.

S. G. P. and K. J. are supported by the NKRDP (National Key Research and Development Program) under Grant No. 2016YFA0301503 and NSFC under Grants No. 11474315, No. 11674358, No. 11434015, and No. 91336106. S. T. is supported by the U.S. National Science Foundation CAREER award Grant No. PHY-1352208.

S. G. P. and C. X. Z. contributed equally to this work.

\*shina.tan@physics.gatech.edu

†kjjiang@wipm.ac.cn

- [1] S. Tan, *Ann. Phys. (Amsterdam)* **323**, 2952 (2008); S. Tan, *Ann. Phys. (Amsterdam)* **323**, 2971 (2008); **323**, 2987 (2008).
- [2] W. Zwerger, *The BCS-BEC Crossover and the Unitary Fermi Gas*, Col 836 of Lecture Notes in Physics (Springer, Berlin, 2011); please see Chap. 6 for a brief review.
- [3] Z. Yu, J. H. Thywissen, and S. Zhang, *Phys. Rev. Lett.* **115**, 135304 (2015); **117**, 019901(E) (2016).
- [4] M. Y. He, S. L. Zhang, H. M. Chan, and Q. Zhou, *Phys. Rev. Lett.* **116**, 045301 (2016).
- [5] C. Luciuk, S. Trotzky, S. Smale, Z. Yu, S. Zhang, and J. H. Thywissen, *Nat. Phys.* **12**, 599 (2016).
- [6] S. G. Peng, X. J. Liu, and H. Hu, *Phys. Rev. A* **94**, 063651 (2016).
- [7] R. J. Fletcher, R. Lopes, J. Man, N. Navon, R. P. Smith, M. W. Zwierlein, and Z. Hadzibabic, *Science* **355**, 377 (2017).
- [8] Y. J. Lin, K. Jimenez-Garcia, and I. B. Spielman, *Nature (London)* **471**, 83 (2011).
- [9] P. J. Wang, Z. Q. Yu, Z. K. Fu, J. Miao, L. H. Huang, S. J. Chai, H. Zhai, and J. Zhang, *Phys. Rev. Lett.* **109**, 095301 (2012).
- [10] L. W. Cheuk, A. T. Sommer, Z. Hadzibabic, T. Yefsah, W. S. Bakr, and M. W. Zwierlein, *Phys. Rev. Lett.* **109**, 095302 (2012).
- [11] X. L. Qi and S. C. Zhang, *Phys. Today* **63**, 33 (2010).
- [12] M. Z. Hasan and C. L. Kane, *Rev. Mod. Phys.* **82**, 3045 (2010).
- [13] J. Dalibard, F. Gerbier, G. Juzeliunas, and P. Ohberg, *Rev. Mod. Phys.* **83**, 1523 (2011).
- [14] J. Zhang, H. Hu, X.-J. Liu, and H. Pu, *Annual Review of Cold Atoms and Molecules* (World Scientific, Singapore, 2014), Vol. 2, p. 81.
- [15] B. M. Anderson, G. Juzeliunas, V. M. Galitski, and I. B. Spielman, *Phys. Rev. Lett.* **108**, 235301 (2012).
- [16] X. L. Cui, *Phys. Rev. A* **85**, 022705 (2012).
- [17] Y. X. Wu and Z. H. Yu, *Phys. Rev. A* **87**, 032703 (2013).
- [18] X. L. Cui, *Phys. Rev. A* **95**, 030701 (2017).
- [19] P. Zhang, L. Zhang, and Y. J. Deng, *Phys. Rev. A* **86**, 053608 (2012).
- [20] P. J. Wang, Z. Q. Yu, Z. K. Fu, J. Miao, L. H. Huang, S. J. Chai, H. Zhai, and J. Zhang, *Phys. Rev. Lett.* **109**, 095301 (2012); L. W. Cheuk, A. T. Sommer, Z. Hadzibabic, T. Yefsah, W. S. Bakr, and M. W. Zwierlein, *Phys. Rev. Lett.* **109**, 095302 (2012).
- [21] J. Zhang, E. G. M. van Kempen, T. Bourdel, L. Khaykovich, J. Cubizolles, F. Chevy, M. Teichmann, L. Tarruell, S. J. J. M. F. Kokkelmans, and C. Salomon, *Phys. Rev. A* **70**, 030702 (2004); C. H. Schunck, M. W. Zwierlein, C. A. Stan, S. M. F. Raupach, W. Ketterle, A. Simoni, E. Tiesinga, C. J. Williams, and P. S. Julienne, *Phys. Rev. A* **71**, 045601 (2005).
- [22] For the  $s$ -wave interaction, there is a difference of the factor  $8\pi$  from the well-known form of adiabatic relations [see Eq. (36)–(37) of Ref. [6]]. This is because here we include the spherical harmonics  $Y_{00}(\hat{\mathbf{r}}) = 1/\sqrt{4\pi}$  in the  $s$ -partial wave function. Besides, an additional factor  $1/2$  is introduced in order to keep the definition of the contacts consistent with those in the tail of the momentum distribution at large  $\mathbf{q}$ .
- [23] L. D. Landau and E. M. Lifshitz, *Statistical Physics*, 3rd ed. (Elsevier, Singapore, 2007), Part I.
- [24] E. Braaten and L. Platter, *Phys. Rev. Lett.* **100**, 205301 (2008); E. Braaten, D. Kang, and L. Platter, *Phys. Rev. A* **78**, 053606 (2008).
- [25] M. Barth and W. Zwerger, *Ann. Phys. (Amsterdam)* **326**, 2544 (2011).