

Two-channel-model description of confinement-induced Feshbach moleculesShi-Guo Peng,¹ Hui Hu,² Xia-Ji Liu,² and Kaijun Jiang^{1,3,*}¹*State Key Laboratory of Magnetic Resonance and Atomic and Molecular Physics, Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, Wuhan 430071, China*²*ARC Centre of Excellence for Quantum-Atom Optics, Centre for Atom Optics and Ultrafast Spectroscopy, Swinburne University of Technology, Melbourne 3122, Australia*³*Center for Cold Atom Physics, Chinese Academy of Sciences, Wuhan 430071, China*

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Using a two-channel model, we investigate theoretically the binding energy of confinement-induced Feshbach molecules in two- and one-dimensional ultracold atomic systems near a Feshbach resonance. We show that the two-channel prediction will evidently deviate from the simple single-channel theory as the width of Feshbach resonances decreases. For the one-dimensional system, we perform a full two-channel calculation, with the inclusion of bare interatomic interactions in the open channel. Away from the resonance, we find a sizable correction to the binding energy, if we neglect the bare interatomic interactions as in the previous work [D. B. M. Dickerscheid and H. T. C. Stoof, *Phys. Rev. A* **72**, 053625 (2005)]. We compare our theoretical results with the existing experimental data and present predictions for narrow Feshbach resonances that could be tested in future experiments.

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I. INTRODUCTION

The confinement-induced resonance (CIR) is one of the most intriguing phenomena in low-dimensional ultracold atomic systems and has attracted a great deal of interest. It was first predicted by Olshanii using a single-channel model in 1998, when considering two-body collisions in one-dimensional (1D) harmonic waveguides [1]. Later, this study was extended to two dimensions by Petrov and coworkers in 2000 [2,3]. To date, there are a number of experimental confirmations of the existence of CIR in both 1D and two-dimensional (2D) setups.

To confirm experimentally CIR, it is convenient to use a Feshbach resonance to tune the interatomic interactions [4,5] and to measure the binding energy of the resulting confinement-induced Feshbach molecules (CIFMs). The existence of CIFMs in a 1D Fermi gas of ⁴⁰K atoms was first detected by Moritz *et al.* [6]. Using CIR in a 1D Bose gas of ¹³³Cs atoms [7], Haller *et al.* realized the crossover from the Tonks-Girardeau [8,9] to super-Tonks-Girardeau regime [10]. Interestingly, in such 1D experiments [11], an anomalous splitting of the CIRs appeared when an anisotropy in the transverse confinement was introduced, which could not be explained by generalizing Olshanii's theory to anisotropic transverse confinement [12]. But soon it was realized that the splitting of the CIRs resulted from the coupling between the center-of-mass (c.m.) and relative motions of two incoming atoms due to the anharmonicity of the trap [13,14]. Besides these 1D experiments, CIR was also recently observed in a 2D Fermi gas of ⁴⁰K atoms [15]. The interaction between atoms was tuned by using a magnetic Feshbach resonance at $B_0 = 224.2$ G, and the binding energy of the CIFMs was measured. Compared with the prediction of the single-channel theory [2,3], however, the observed binding energy is larger than the theoretical prediction by 4 kHz in magnitude [15]. This

discrepancy may be understood from the picture of fermionic polaron, as suggested by Schmidt *et al.* [16]. Alternatively, the discrepancy may also be solved by introducing an energy-dependent interaction, as shown experimentally for ⁴⁰K atoms near another Feshbach resonance $B_0 = 202.1$ G [17].

Here we aim to study CIR and CIFMs by using a two-channel model, with the inclusion of the effect of bound molecule states in the closed channel. This is a physically more realistic description for Feshbach resonances and therefore should provide a better description for CIFMs which are measured experimentally. It is well known that the two-channel model yields the same theoretical predictions as the single-channel model in the limit of broad Feshbach resonances [18,19]. The experimentally utilized resonances for ⁴⁰K and ¹³³Cs atoms are broad. As a result, our two-channel model description may not give an improved understanding of the existing CIR measurements. However, our two-channel results should provide a useful guide for future CIR experiments on relatively narrow Feshbach resonances.

We note that a two-channel calculation was previously carried out for 1D CIFMs by Dickerscheid and Stoof in 2005 [20]. However, in their study the bare interatomic interactions were neglected. It was shown then the two-channel model gives a better agreement with the 1D experimental data than the single-channel model [20].

In this work, we present a two-channel calculation for the binding energy of CIFMs in 2D systems under an axially harmonic confinement. Near the Feshbach resonance, the channel coupling will dominate the contribution to the binding energy of the dressed molecules, which allows us to neglect the bare interatomic interactions in the open channel. At this resonance limit, we find that the two-channel result coincides with the single-channel theory for a broad Feshbach resonance, but will evidently deviate from the single-channel theory as the resonance width decreases. We compare our theoretical calculation with the recent 2D ⁴⁰K experiment [17], in which a broad Feshbach resonance at $B_0 = 202.1$ G is used to control the interatomic interactions. We find that the two-channel

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theory is in agreement with the single-channel result, as expected, and both agree well with the experimental data at the resonance. To demonstrate the in-equivalence between the two-channel and single-channel theories, we predict the binding energy of 2D CIFMs for two relatively narrow resonances, ^{23}Na at $B_0 = 907$ G and ^{87}Rb at $B_0 = 1007.4$ G.

We also perform a *full* two-channel calculation for 1D CIFMs. We find that the bare interatomic interactions will give rise to a sizable correction to the binding energy *away* from Feshbach resonances. This bare interatomic interaction was neglected in the previous treatment [20]. By restoring the bare interatomic interactions, the two-channel result coincides with the single-channel prediction, as we anticipate for a broad resonance. Thus, we realize that the previous better agreement reported by Dickerscheid and Stoof is not convincing.

The paper is arranged as follows. We first present the two-channel Hamiltonian in Sec. II. Then the ansatz of two-body wave functions for 2D and 1D systems is constructed in Sec. III. In Sec. IV, we solve the two-body problems and calculate the binding energy of CIFMs. In Sec. V, we report the binding energy as a function of the width of Feshbach resonances and show how the two-channel result deviates from the single-channel prediction as the resonance width decreases. We also compare our theoretical results with the recent experiments and discuss in detail the correction of the bare interatomic interactions to the binding energy of 1D CIFMs. Finally, our main results are summarized in Sec. VI.

II. TWO-CHANNEL HAMILTONIAN

For a two-component Fermi gas with atomic mass m , the two-channel effective Hamiltonian that we consider includes the following single-particle Hamiltonian and interaction Hamiltonian [21,22]

$$\begin{aligned} \mathcal{H}_0 = & \sum_{\sigma} \int d^3\mathbf{r} \hat{\psi}_{\sigma}^{\dagger}(\mathbf{r}) \left[-\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(\mathbf{r}) \right] \hat{\psi}_{\sigma}(\mathbf{r}) \\ & + \int d^3\mathbf{r} \hat{\Psi}^{\dagger}(\mathbf{r}) \left[-\frac{\hbar^2}{4m} \nabla^2 + 2V_{\text{ext}}(\mathbf{r}) + \Delta(B) \right] \hat{\Psi}(\mathbf{r}) \end{aligned} \quad (1)$$

and

$$\begin{aligned} \mathcal{H}_{\text{int}} = & U \int d^3\mathbf{r} \hat{\psi}_{\uparrow}^{\dagger}(\mathbf{r}) \hat{\psi}_{\downarrow}^{\dagger}(\mathbf{r}) \hat{\psi}_{\downarrow}(\mathbf{r}) \hat{\psi}_{\uparrow}(\mathbf{r}) \\ & + g \int d^3\mathbf{r} \{ \hat{\Psi}^{\dagger}(\mathbf{r}) \hat{\psi}_{\downarrow}(\mathbf{r}) \hat{\psi}_{\uparrow}(\mathbf{r}) + \text{H.c.} \}. \end{aligned} \quad (2)$$

Here, $\hat{\psi}_{\sigma}(\mathbf{r})$ ($\sigma = \uparrow, \downarrow$) and $\hat{\Psi}(\mathbf{r})$ are the field operators of fermionic atoms in the open (atomic) channel and of bosonic molecules in the closed (molecular) channel, respectively. $V_{\text{ext}}(\mathbf{r})$ is the external trapping potential, which could be described by a harmonic trap as a good approximation. To realize a 2D system, a tight axial confinement is applied experimentally, while the radial (transverse) confinement is much weaker. Thus, the radial motion of atoms is approximately free. Theoretically, we can only consider a tight axial confinement $V_{\text{ext}}(z) = m\omega_{\parallel}^2 z^2/2$ along the z direction. In a like manner, for a 1D system, the external trapping potential can be chosen as $V_{\text{ext}}(\rho) = m\omega_{\perp}^2 \rho^2/2$, which is in x - y plane with

$\rho^2 = x^2 + y^2$. $\Delta(B)$ is the detuning between the atomic and molecular channels, which could be tuned by a magnetic field near a Feshbach resonance. U describes the bare interaction between atoms with different spins in the atomic channel, and g denotes the coupling strength between the atomic and molecular channels. We have used the constants U and g to describe the coupling processes both for the bare interatomic interaction and the formation of molecules, which is analogous to those used in the pseudopotential theory. Consequently, a divergence in high energy will appear. Thus, a regularization must be introduced as we shall mention later.

III. ANSATZ OF THE TWO-BODY WAVE FUNCTION

To calculate the binding energy of CIFMs, let us solve a two-body problem. In this paper, we are only interested in the tight-confinement limit $\hbar\omega_{\parallel,\perp} \gg k_B T$, where k_B is the Boltzmann constant and T is the temperature. For an ultracold Fermi gas with a pretty low temperature, almost all the atoms occupy the lowest-energy (ground) state of the confined direction and cannot be excited to the higher energy levels during collisions. Due to the harmonic confinement, the c.m. motion is completely decoupled from the relative motion of two atoms. Therefore, the c.m. of the two atoms will always stay in the ground state of the confined direction. In addition, there is always a global translation invariance in the untrapped directions, which means the c.m. momentum \mathbf{K} in these directions is a good quantum number. Thus, an ansatz of the two-body wave function could take the following form

$$\begin{aligned} |\varphi_2\rangle = & \left[\int d^3\mathbf{r} e^{i\mathbf{K}\cdot\rho} \phi_0(z) \hat{\Psi}^{\dagger}(\mathbf{r}) \right. \\ & + \int d^3\mathbf{r}_1 d^3\mathbf{r}_2 e^{i\mathbf{K}\cdot(\rho_1+\rho_2)/2} \phi_0\left(\frac{z_1+z_2}{2}\right) \\ & \left. \times \mathcal{Q}_2(\mathbf{r}_1 - \mathbf{r}_2) \hat{\psi}_{\uparrow}^{\dagger}(\mathbf{r}_1) \hat{\psi}_{\downarrow}^{\dagger}(\mathbf{r}_2) \right] |0\rangle \end{aligned} \quad (3)$$

for a 2D system and

$$\begin{aligned} |\varphi_1\rangle = & \left[\int d^3\mathbf{r} e^{iKz} \phi_{00}(\rho) \hat{\Psi}^{\dagger}(\mathbf{r}) \right. \\ & + \int d^3\mathbf{r}_1 d^3\mathbf{r}_2 e^{iK(z_1+z_2)/2} \phi_{00}\left(\frac{\rho_1+\rho_2}{2}\right) \\ & \left. \times \mathcal{Q}_1(\mathbf{r}_1 - \mathbf{r}_2) \hat{\psi}_{\uparrow}^{\dagger}(\mathbf{r}_1) \hat{\psi}_{\downarrow}^{\dagger}(\mathbf{r}_2) \right] |0\rangle \end{aligned} \quad (4)$$

for a 1D system. Here $|0\rangle$ stands for the vacuum state and $\mathcal{Q}_D(\mathbf{r})$ is the relative wave function of the two atoms in a “ D ”-dimensional system, which is spatially antisymmetric [e.g., $\mathcal{Q}_2(-\mathbf{r}) = -\mathcal{Q}_2(\mathbf{r})$]. $\phi_{00}(\rho)$ and $\phi_0(z)$ are the ground states of 1D and 2D harmonic oscillators, respectively.

IV. SOLUTION OF TWO-BODY PROBLEM

A. In a 2D system

Let us first consider an axially confined Fermi gas, where atoms can move freely in the x - y plane. By acting the two-channel Hamiltonian (1) and (2) onto the two-body wave-function ansatz (3), the Schrödinger equation

$\mathcal{H}|\varphi_2\rangle = \varepsilon|\varphi_2\rangle$ is equivalent to the following set of equations (see the Appendix for details)

$$\left(-\frac{\hbar^2}{2\mu}\nabla^2 + \frac{1}{2}\mu\omega_{\parallel}^2 z^2\right) Q_2(\mathbf{r}) + [U Q_2(\mathbf{r}) + g]\delta(\mathbf{r}) = E Q_2(\mathbf{r}), \quad (5)$$

$$E = \Delta(B) + g Q_2(0), \quad (6)$$

where the c.m. part has been separated out. $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ is the relative coordinate, $\mu = m/2$ is the reduced mass, and $E = \varepsilon - \hbar^2 K^2/4m - \hbar\omega_{\parallel}/2$ is the relative energy. From Eq. (5), the bound-state solution can be written as

$$Q_2(\mathbf{r}) = -[U Q_2(0) + g]G_E^{(2)}(\mathbf{r}, 0), \quad (7)$$

where $G_E^{(2)}(\mathbf{r}, \mathbf{r}')$ is the Green's function that satisfies

$$\left(-\frac{\hbar^2}{2\mu}\nabla^2 + \frac{1}{2}\mu\omega_{\parallel}^2 z^2 - E\right) G_E^{(2)}(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'). \quad (8)$$

After some straightforward algebra [13], the Green's function $G_E^{(2)}(\mathbf{r}, 0)$ is easily obtained,

$$G_E^{(2)}(\mathbf{r}, 0) = \frac{e^{-z^2/2d_{\parallel}^2}}{2\pi^{3/2}d_{\parallel}^3\hbar\omega_{\parallel}} \int_0^{\infty} dt \frac{\exp\left(\frac{\epsilon}{2}t - \frac{e^{-t}}{1-e^{-t}} \frac{z^2}{d_{\parallel}^2} - \frac{1}{t} \frac{\rho^2}{d_{\parallel}^2}\right)}{t\sqrt{1-e^{-t}}}, \quad (9)$$

where $d_{\parallel} = \sqrt{\hbar/\mu\omega_{\parallel}}$ is the harmonic length and $\epsilon = E/\hbar\omega_{\parallel} - 1/2$. This integral representation of the Green's function (9) is valid only for $\epsilon < 0$. Obviously when $r \rightarrow 0$, the Green's function diverges as

$$\begin{aligned} \lim_{r \rightarrow 0} G_E^{(2)}(\mathbf{r}, 0) &\approx \frac{1}{2\pi^{3/2}d_{\parallel}^3\hbar\omega_{\parallel}} \int_0^{\infty} dt \frac{\exp\left(-\frac{1}{t} \frac{r^2}{d_{\parallel}^2}\right)}{t^{3/2}} \\ &= \frac{1}{2\pi d_{\parallel}^2\hbar\omega_{\parallel}} \frac{1}{r}, \end{aligned} \quad (10)$$

which results in an energy divergence. This is what we have anticipated since we use a pseudopotential method to describe the coupling process both for the interatomic interactions and for the formation of molecules.

To eliminate this singularity, the Green's function must be regularized as follows

$$\begin{aligned} \lim_{r \rightarrow 0} G_{\epsilon}^{(2)}(\mathbf{r}, 0) &= \lim_{r \rightarrow 0} \left[G_E^{(2)}(\mathbf{r}, 0) - \frac{1}{2\pi d_{\parallel}^2\hbar\omega_{\parallel}} \frac{1}{r} \right], \\ &= \frac{1}{2\pi^{3/2}d_{\parallel}^3\hbar\omega_{\parallel}} \mathcal{F}_2(\epsilon), \end{aligned} \quad (11)$$

where

$$\mathcal{F}_2(\epsilon) = \int_0^{\infty} dt \left[\frac{\exp(\epsilon t/2)}{t\sqrt{1-e^{-t}}} - \frac{1}{t^{3/2}} \right]. \quad (12)$$

Then, by substituting Eqs. (11) and (12) into Eq. (7), $Q_2(0)$ can be easily solved

$$Q_2(0) = -g[2\pi^{3/2}d_{\parallel}^3\hbar\omega_{\parallel}\mathcal{F}_2^{-1}(\epsilon) + U]^{-1}. \quad (13)$$

Consequently, by combining Eqs. (6) and (13), the binding energy E_B of 2D CIFMs should satisfy the following

self-consistent equation

$$E_B = \Delta(B) - g^2[2\pi^{3/2}d_{\parallel}^3\hbar\omega_{\parallel}\mathcal{F}_2^{-1}(E_B/\hbar\omega_{\parallel} - 1/2) + U]^{-1}. \quad (14)$$

By solving this self-consistent equation the binding energy can be obtained.

B. In a 1D system

For a 1D system, atoms can only move freely along the axial direction, while the radial motion is frozen. Operating on $|\varphi_1\rangle$, Eq. (4), with the Hamiltonian (1) and (2), the Schrödinger equation is deduced to the following set of equations

$$\left(-\frac{\hbar^2}{2\mu}\nabla^2 + \frac{1}{2}\mu\omega_{\perp}^2 \rho^2\right) Q_1(\mathbf{r}) + [U Q_1(\mathbf{r}) + g]\delta(\mathbf{r}) = E Q_1(\mathbf{r}), \quad (15)$$

$$E = \Delta(B) + g Q_1(0), \quad (16)$$

where the relative energy is $E = \varepsilon - \hbar^2 K^2/4m - \hbar\omega_{\perp}$. After the similar derivation as that for the 2D case, a self-consistent equation of the binding energy E_B can easily be obtained

$$E_B = \Delta(B) - g^2[2\pi^{3/2}d_{\perp}^3\hbar\omega_{\perp}\mathcal{F}_1^{-1}(E_B/\hbar\omega_{\perp} - 1) + U]^{-1}, \quad (17)$$

where $d_{\perp} = \sqrt{\hbar/\mu\omega_{\perp}}$ and

$$\mathcal{F}_1(\epsilon) = \int_0^{\infty} dt \left[\frac{\exp(\epsilon t/2)}{\sqrt{t}(1-e^{-t})} - \frac{1}{t^{3/2}} \right]. \quad (18)$$

Then the binding energy of the 1D CIFMs can be solved using the self-consistent equation (17).

V. RESULTS AND DISCUSSION

A. At the resonance

The single-channel theory is used to describe the universality for a strongly interacting Fermi gas near a broad Feshbach resonance [23,24], in which all other length scales become irrelevant except the average interatomic distance. However, there is always an intrinsic length r_* for all Feshbach resonances, which is related to the resonance width. It can be defined as Ref. [18]

$$r_* = \frac{\hbar^2}{2ma_{\text{bg}}\Delta\mu\Delta B}, \quad (19)$$

where a_{bg} is the background scattering length in the open channel, ΔB is the width of the resonance, and $\Delta\mu$ is the magnetic moment difference of the molecular state with respect to the threshold of two free atoms. For a broad resonance, this intrinsic length is quite small due to the large resonance width, thus it is reasonable that the single-channel theory gives a good description of Feshbach resonances. However, when the resonance width becomes narrow, the intrinsic length r_* cannot be ignored anymore, and will obviously affect the properties of the resonance. Consequently, the prediction of the two-channel theory, which is physically more realistic, is expected to deviate from that of the single-channel theory. To demonstrate this deviation, let us focus on the situation right at the resonance, where the bare interatomic interaction U could

be reasonably neglected, as we shall discuss in greater detail later. The channel coupling strength g is given by Refs. [20,25],

$$g = \hbar \sqrt{\frac{4\pi a_{\text{bg}} \Delta B \Delta \mu}{m}} = \frac{\hbar^2}{m} \sqrt{\frac{2\pi}{r_*}}. \quad (20)$$

In combination with Eq. (14), the binding energy of 2D CIFMs predicted by the two-channel theory at the resonance can be written as

$$\frac{\epsilon_B}{\hbar\omega_{\parallel}} = -\frac{d_{\parallel}}{4\sqrt{\pi}r_*} \mathcal{F}_2\left(\frac{\epsilon_B}{\hbar\omega_{\parallel}}\right) - \frac{1}{2}, \quad (21)$$

where $\epsilon_B = E_B - \hbar\omega_{\parallel}/2$.

The binding energy $\epsilon_B/(\hbar\omega_{\parallel})$ of 2D CIFMs as a function of the intrinsic length d_{\parallel}/r_* is shown in Fig. 1. At a broad resonance limit (i.e., $d_{\parallel}/r_* \rightarrow \infty$) the two-channel prediction approaches to the result of the single-channel theory, which is predicted to be $-0.244\hbar\omega_{\parallel}$ [26]. However, as the resonance becomes narrow, the two-channel prediction will obviously deviate from the single-channel result, and the binding energy of 2D CIFMs is strongly dependent on the intrinsic length or the resonance width. In the recent 2D ^{40}K experiment [17], the binding energy of 2D CIFMs is measured across a broad Feshbach resonance at $B_0 = 202.1$ G with $d_{\parallel}/r_* \sim 110$. The observed binding energy near the resonance is indicated in Fig. 1 by the (red) solid circles with error bar. Since there are no experimental data exactly on the resonance, two points near the resonance are chosen, and the binding energy right at the resonance should lie between these two points.

We also predict the binding energy of 2D CIFMs for two relatively narrow resonances of ^{23}Na at $B_0 = 907$ G and ^{87}Rb at $B_0 = 1007.4$ G, which are presented by the (blue) circle and (wine) triangle in Fig. 1, respectively.

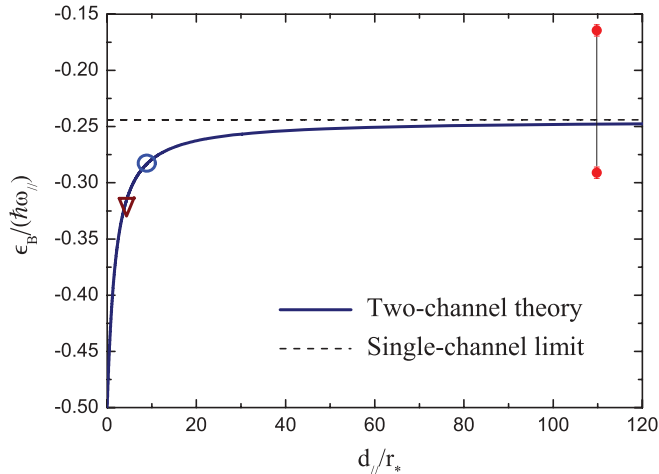


FIG. 1. (Color online) The binding energy of 2D CIFMs as a function of d_{\parallel}/r_* right at the Feshbach resonance. The (red) solid circles with error bar are taken from the 2D ^{40}K experiment [17] near the Feshbach resonance at $B_0 = 202.1$ G. The two-channel predictions for the binding energy of two relatively narrow resonances for ^{23}Na at $B_0 = 907$ G (blue circle) and ^{87}Rb at $B_0 = 1007.4$ G (wine triangle) are also presented. Here, the trap frequency ω_{\parallel} is chosen to be $2\pi \times 75$ kHz as in Ref. [17].

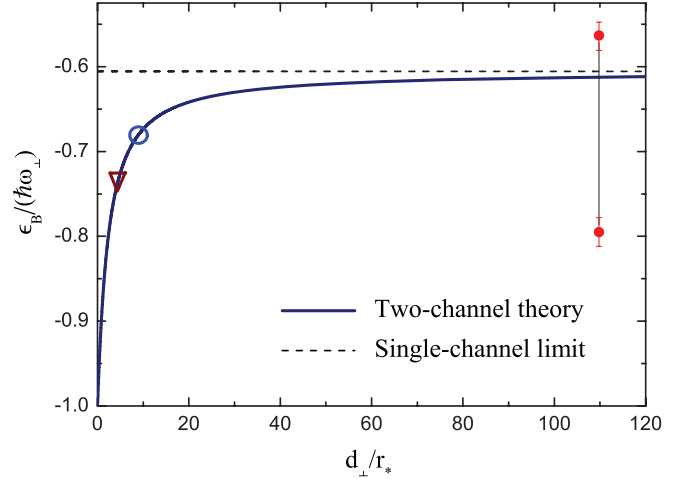


FIG. 2. (Color online) The same as Fig. 1 but for 1D CIFMs. The (red) solid circles with error bar are taken from the 1D ^{40}K experiment [6], showing the two data points nearest to the Feshbach resonance. Here, the trap frequency ω_{\perp} is chosen to be $2\pi \times 69$ kHz as in Ref. [6].

In a like manner, the binding energy of 1D CIFMs at the resonance is given by

$$\frac{\epsilon_B}{\hbar\omega_{\perp}} = -\frac{d_{\perp}}{4\sqrt{\pi}r_*} \mathcal{F}_1\left(\frac{\epsilon_B}{\hbar\omega_{\perp}}\right) - 1, \quad (22)$$

where $\epsilon_B = E - \hbar\omega_{\perp}$. The binding energy of 1D CIFMs as a function of the intrinsic length is plotted in Fig. 2.

B. 2D CIFMs across the Feshbach resonance

In the following, let us investigate the binding energy of 2D CIFMs across Feshbach resonances. By using Eqs. (12) and (14), the binding energy predicted by the two-channel theory satisfies

$$\epsilon_B + \frac{1}{2}\hbar\omega_{\parallel} = \Delta(B) - g^2 \left[2\pi^{3/2} d_{\parallel}^3 \hbar\omega_{\parallel} \mathcal{F}_2^{-1}\left(\frac{\epsilon_B}{\hbar\omega_{\parallel}}\right) + U \right]^{-1}. \quad (23)$$

We recall that the binding energy given by the single-channel theory is determined by the following equation [3,26]

$$\frac{d}{a_{3\text{D}}} = -\frac{1}{\sqrt{\pi}} \mathcal{F}_2\left(\frac{\epsilon_B}{\hbar\omega_{\parallel}}\right), \quad (24)$$

where $a_{3\text{D}}$ is the effective three-dimensional (3D) s -wave scattering length.

In a magnetic Feshbach resonance, the detuning between the open and closed channels $\Delta(B) = \Delta\mu(B - B_0)$ varies as a function of the magnetic field B and depends on the moment difference $\Delta\mu$ between the open and closed channels. The bare interatomic interaction U and the channel-coupling constant g are given by $U_0 = 4\pi\hbar^2 a_{\text{bg}}/m$ and Eq. (20), respectively [20,25]. The effective 3D scattering length $a_{3\text{D}}$ is adjusted by the magnetic field B according to

$$a_{3\text{D}}(B) = a_{\text{bg}} \left(1 - \frac{\Delta B}{B - B_0} \right). \quad (25)$$

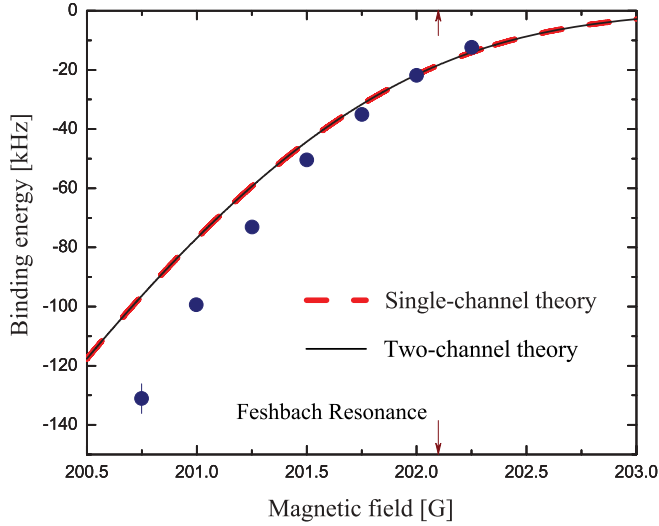


FIG. 3. (Color online) The binding energy of 2D CIFMs across the Feshbach resonance at $B_0 = 202.1$ G, compared with the experimental data (solid circles) [17]. The parameters for this resonance are $\Delta B = 7.8$ G, $\Delta\mu = 1.68\mu_B$, $a_{bg} = 174a_B$, where μ_B and a_B are the Bohr magneton and radius, respectively. The trap frequency in the strongly confined direction is $\omega_{\parallel} = 2\pi \times 75$ kHz.

The two-channel prediction of the binding energy of 2D CIFMs across a *broad* Feshbach resonance as well as that of the single-channel theory is illustrated in Fig. 3, in comparison with the recent 2D ^{40}K experiment [17]. It can be seen clearly that the two-channel theory agrees with the single-channel result for such a broad resonance, and both agree well with the experiment at the resonance. However, away from the resonance, the observed binding energy deviates from both two-channel and single-channel predictions.

C. 1D CIFMs across the Feshbach resonance

We now turn to the binding energy of 1D CIFMs. According to Eqs. (17) and (18), the two-channel prediction for the binding energy of 1D CIFMs is given by

$$\epsilon_B + \hbar\omega_{\perp} = \Delta(B) - g^2 \left[2\pi^{3/2} d_{\perp}^3 \hbar\omega_{\perp} \times \mathcal{F}_1^{-1} \left(\frac{\epsilon_B}{\hbar\omega_{\perp}} \right) + U \right]^{-1}. \quad (26)$$

The two-channel result is plotted as well as the single-channel prediction in Fig. 4. We find that the results of the two-channel and single-channel theories are almost the same, and both agree with the experiment [6] at the resonance. This agreement is anticipated since the resonance used by the authors of Ref. [6] is also very broad, and the two-channel theory should approach the single-channel result, as we have already seen in Fig. 2.

In Ref. [20], Dickerscheid and Stoof suggested ignoring the bare interatomic interaction U , by assuming that this background interaction is quite small compared to the channel coupling near the resonance. Under this approximation, Eq. (26) yields

$$\epsilon_B + \hbar\omega_{\perp} \approx \Delta(B) - \frac{g^2}{2\pi^{3/2} d_{\perp}^3 \hbar\omega_{\perp}} \mathcal{F}_1 \left(\frac{\epsilon_B}{\hbar\omega_{\perp}} \right), \quad (27)$$

which recovers the results in Ref. [20]. Here, we define a different harmonic length $d_{\perp} = \sqrt{\hbar/\mu\omega_{\perp}}$, which is $\sqrt{2}$ times larger than that in Ref. [20], and $\mathcal{F}_1(\epsilon) = \sqrt{\pi}\zeta(1/2, -\epsilon)$, where $\zeta(s, q)$ is the Hurwitz ζ function. The results of Eq. (27) are plotted in Fig. 4 by a (blue) dash-dotted curve. It seems that the two-channel theory provides an improved agreement with the experimental data, compared with the single-channel theory. Here, however, we find that if the bare interatomic interaction is correctly included, the result of the *full* two-channel theory returns back to that of the single-channel calculation, as expected for a *broad* Feshbach resonance.

Now, it is interesting to discuss the condition under which the bare interatomic interaction U could be neglected. From Eq. (26), we find whether the bare interatomic interaction could be neglected lies on the competition between $2\pi^{3/2} d_{\perp}^3 \hbar\omega_{\perp} \mathcal{F}_1^{-1}[\epsilon_B/(\hbar\omega_{\perp})]$ and U , rather than the ratio of g/U . Thus, the condition can be taken as

$$2\pi^{3/2} d_{\perp}^3 \hbar\omega_{\perp} \mathcal{F}_1^{-1} \left(\frac{\epsilon_B}{\hbar\omega_{\perp}} \right) \gg U, \quad (28)$$

or

$$\left| \mathcal{F}_1 \left(\frac{\epsilon_B}{\hbar\omega_{\perp}} \right) \right| \ll \sqrt{\pi} \frac{d_{\perp}}{a_{bg}}. \quad (29)$$

The function $\mathcal{F}_1[\epsilon_B/(\hbar\omega_{\perp})]$ as a function of the magnetic field is presented in Fig. 5. Obviously, the condition (29) can be fulfilled well very close to the resonance, which means ignoring the bare interatomic interaction is indeed a good approximation at the resonance limit. However, away from the resonance, we see that the bare interatomic interaction will bring a sizable correction to the binding energy of the

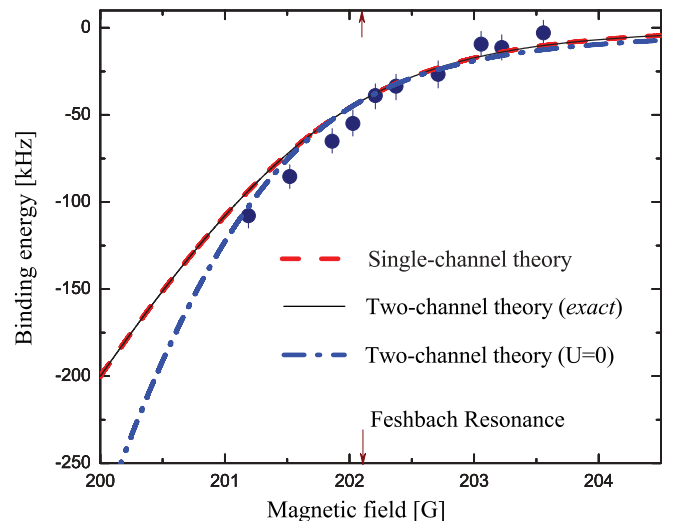


FIG. 4. (Color online) The binding energy of 1D CIFMs across the Feshbach resonance at $B_0 = 202.1$ G, compared with the experimental data (solid circles) [6]. The (blue) dash-dotted curve is the result of the authors of Ref. [20], obtained by ignoring the bare interatomic interaction in the open channel (i.e., $U = 0$). Here, the trap frequency in the strongly confined direction is $\omega_{\perp} = 2\pi \times 69$ kHz [6].

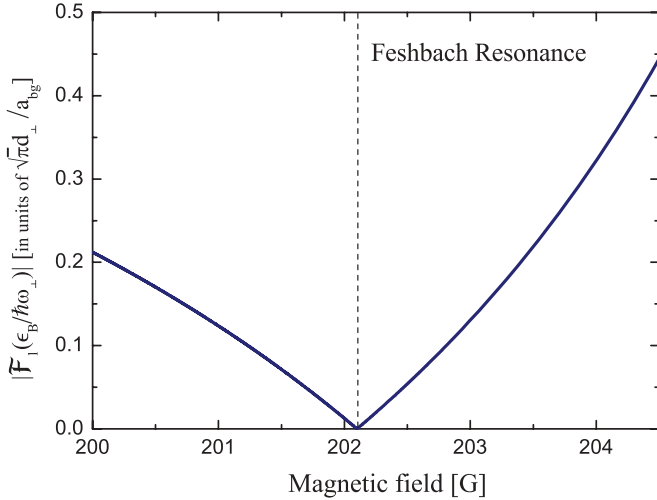


FIG. 5. (Color online) The function $\mathcal{F}_1[\epsilon_B/(\hbar\omega_\perp)]$, in units of $\sqrt{\pi}d_\perp/a_{bg}$, varies as a function of the magnetic field for a 1D system across the resonance $B_0 = 202.1$ G. The dashed line indicates the location of the Feshbach resonance.

molecules, which makes the approximation results deviate from those of the full two-channel calculations, as we have already seen in Fig. 4.

VI. CONCLUSION

In summary, we have presented a two-channel theory for calculating the binding energy of confinement-induced Feshbach molecules in 2D and 1D ultracold atomic systems. We have found that the two-channel results will deviate from the single-channel predictions as the width of Feshbach resonances decreases, although these two theories give the same results in the limit of broad resonances. Compared with the recent experiments, where a *broad* magnetic Feshbach resonance was used to control the interatomic interactions,

both the two-channel and single-channel theories agree well with the data near resonance.

For 1D confinement-induced Feshbach molecules, a full two-channel calculation has been performed. Compared with the previous work [20], in which the background interatomic interaction in the open channel was neglected, we have found that the bare interatomic interaction will yield a sizable correction to the binding energy away from the resonance. Thus, in contrast to the difference shown in Ref. [20], the two-channel result agrees very well with the single-channel prediction, as we may anticipate for a *broad* Feshbach resonance.

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APPENDIX: THE SCHRÖDINGER EQUATION FOR A 2D SYSTEM

In this Appendix, we present the detailed derivation of how the 2D Schrödinger equation $\mathcal{H}|\varphi_2\rangle = \varepsilon|\varphi_2\rangle$ is reduced to the coupled equations (5) and (6). By using the commutation and anticommutation relations of the Bose and Fermi field operators

$$[\hat{\Psi}(\mathbf{r}), \hat{\Psi}^\dagger(\mathbf{r}')]_- = \delta(\mathbf{r} - \mathbf{r}'), \quad (\text{A1})$$

and

$$[\hat{\psi}_\sigma(\mathbf{r}), \hat{\psi}_{\sigma'}^\dagger(\mathbf{r}')]_+ = \delta(\mathbf{r} - \mathbf{r}')\delta_{\sigma\sigma'}, \quad (\text{A2})$$

where \mp represent the commutation and anticommutation relations, respectively, we obtain

$$\begin{aligned} \mathcal{H}|\varphi_2\rangle = & \int d^3\mathbf{r}_1 d^3\mathbf{r}_2 \hat{\psi}_\uparrow^\dagger(\mathbf{r}_1) \hat{\psi}_\downarrow^\dagger(\mathbf{r}_2) \left\{ \left[-\frac{\hbar^2}{2m} \nabla_{\mathbf{r}_1}^2 + V_{\text{ext}}(\mathbf{r}_1) - \frac{\hbar^2}{2m} \nabla_{\mathbf{r}_2}^2 + V_{\text{ext}}(\mathbf{r}_2) + U_1 \delta(\mathbf{r}_1 - \mathbf{r}_2) \right] \right. \\ & \times e^{i\mathbf{K}\cdot(\rho_1+\rho_2)/2} \phi_0 \left(\frac{z_1+z_2}{2} \right) Q_2(\mathbf{r}_1 - \mathbf{r}_2) + g\delta(\mathbf{r}_1 - \mathbf{r}_2) e^{i\mathbf{K}\cdot(\rho_1+\rho_2)/2} \phi_0 \left(\frac{z_1+z_2}{2} \right) \left. \right\} |0\rangle \\ & + \int d^3\mathbf{r} \hat{\Psi}^\dagger(\mathbf{r}) \left[\frac{\hbar^2 K^2}{4m} + \frac{1}{2}\hbar\omega + \Delta(B) + gQ_2 t(0) \right] e^{i\mathbf{K}\cdot\rho} \phi_0(z) |0\rangle. \end{aligned} \quad (\text{A3})$$

Here we have introduced pseudopotentials to describe the bare interatomic interaction and the channel coupling. Then, by comparing the corresponding terms in $\mathcal{H}|\varphi_2\rangle$ and $\varepsilon|\varphi_2\rangle$, we arrive at

$$\begin{aligned} & \left[-\frac{\hbar^2}{2m} \nabla_{\mathbf{r}_1}^2 + V_{\text{ext}}(\mathbf{r}_1) - \frac{\hbar^2}{2m} \nabla_{\mathbf{r}_2}^2 + V_{\text{ext}}(\mathbf{r}_2) + U\delta(\mathbf{r}_1 - \mathbf{r}_2) \right] e^{i\mathbf{K}\cdot(\rho_1+\rho_2)/2} \phi_0 \left(\frac{z_1+z_2}{2} \right) Q_2(\mathbf{r}_1 - \mathbf{r}_2) \\ & + g\delta(\mathbf{r}_1 - \mathbf{r}_2) e^{i\mathbf{K}\cdot(\rho_1+\rho_2)/2} \phi_0 \left(\frac{z_1+z_2}{2} \right) = \varepsilon e^{i\mathbf{K}\cdot(\rho_1+\rho_2)/2} \phi_0 \left(\frac{z_1+z_2}{2} \right) Q_2(\mathbf{r}_1 - \mathbf{r}_2), \end{aligned} \quad (\text{A4})$$

and

$$\left[\frac{\hbar^2 K^2}{4m} + \frac{1}{2} \hbar \omega_{\parallel} + \Delta(B) + g Q_{2t}(0) \right] e^{i\mathbf{K}\cdot\rho} \phi_0(z) = \varepsilon e^{i\mathbf{K}\cdot\rho} \phi_0(z). \quad (\text{A5})$$

If we separate the c.m. motion from the relative motion, Eqs. (A4) and (A5) yield

$$\left(-\frac{\hbar^2}{2\mu} \nabla^2 + \frac{1}{2} \mu \omega_{\parallel}^2 z^2 \right) Q_2(\mathbf{r}) + [U Q_2(\mathbf{r}) + g] \delta(\mathbf{r}) = E Q_2(\mathbf{r}), \quad (\text{A6})$$

and

$$E = \Delta(B) + g Q_{2t}(0), \quad (\text{A7})$$

respectively, where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, $\mu = m/2$ is the reduced mass and $E = \varepsilon - \hbar^2 K^2/4m - \hbar \omega_{\parallel}/2$ is the relative energy.

In a like manner, we can easily obtain Eqs. (15) and (16) for 1D systems.

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